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Minimum 0-Extensions of Graph Metrics†

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Let $H = (T, U)$ be a connected graph, $V \supseteq T$ a set, and c a non-negative function on the unordered pairs of elements of V. In the *minimum* 0*-extension problem* $(*)$, one is asked to minimize the inner product $c \cdot m$ over all metrics m on V such that (i) m coincides with the distance function of H within T; and (ii) each $v \in V$ is at zero distance from some $s \in T$, i.e. $m(v, s) = 0$.

This problem is known to be NP-hard if $H = K_3$ (as being equivalent to the minimum 3-terminal cut problem), while it is polynomially solvable if $H = K_2$ (the minimum cut problem) or $H = K_{2,r}$ (the minimum $(2, r)$-metric problem). We study problem $(*)$ for all fixed H. More precisely, we consider the linear programming relaxation $(**)$ of $(*)$ that is obtained by dropping condition (ii) above, and call H *minimizable* if the minima in $(*)$ and $(**)$ coincide for all V and c. Note that for such an H problem $(*)$ is solvable in strongly polynomial time.

Our main theorem asserts that H is minimizable if and only if H is bipartite, has no isometric circuit with six or more nodes, and is orientable in the sense that H can be oriented so that nonadjacent edges of any 4-circuit are oppositely directed along this circuit. The proof is based on a combinatorial and topological study of tight and extreme extensions of graph metrics.

Based on the idea of the proof of the NP-hardness for the minimum 3-terminal cut problem in [4], we then show that the minimum 0-extension problem is strongly NP-hard for many non-minimizable graphs H. Other results are also presented.

Key words: Finite metric, Metric extension, Isometric embedding, Multiterminal cut, Multi(commodity) flow.

1. INTRODUCTION

A well-known fact in combinatorial optimization is that for a graph with non-negative weights on the edges, the minimum weight of a cut separating two specified nodes s and t is equal to the minimum weight of a 'fractional cut separating s and t' (cf. [6,7]). This can be reformulated in metric terms by saying that 'every extreme extension of the metric of the graph K_2 is a 0-extension'. Subsequently, a similar property has been established for metrics of some other graphs H, e.g. for any completely bipartite graph $K_{2,r}$ [10]. Can one give a good characterization for the set of graphs H with such a property? This paper answers this question affirmatively. Let us start with some basic definitions.

A *metric* on a set V' is a non-negative real-valued function m that establishes *distances* on pairs of elements of V' satisfying (i) $m(x, x) = 0$, (ii) $m(x, y) = m(y, x)$, and (iii) $m(x, y) + m(y, z) \geq m(x, z)$, for all $x, y, z \in V'$. Zero distances between different elements are allowed (i.e. in fact we deal with semimetrics), and a metric m is called *positive* if $m(x, y) > 0$ for all distinct $x, y \in V'$. Because of (i) and (ii), one may think that m is, in fact, defined on the set $E_{V'}$ of unordered pairs of distinct elements of V'. We usually write xy and $m(xy)$ in place of $\{x, y\}$ and $m(x, y)$, respectively. Sometimes a metric m on V' is denoted by (V', m). For a connected graph $G' = (V', E')$, $\mathrm{d}^{G'}$ denotes its distance function, where $\mathrm{d}^{G'}(xy)$ is the minimum length of a path connecting nodes x and y in G'.

Throughout we deal with a finite set V and a subset $T \subseteq V$ whose elements are called *terminals*. Let μ be a metric on T. A metric m on V is an *extension* of μ to V if $m(st) = \mu(st)$

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for all $s, t \in T$. The set of extensions is described by $O(|V|^3)$ linear constraints; so it forms a polyhedron in $\mathbb{R}^{E_V}$, denoted by $\mathcal{P}_{V,\mu}$. If an extension m obeys the additional requirement that for each $x \in V$, there is $s \in T$ such that $m(xs) = 0$, then m is called a 0-*extension*. Equivalently, let $\{X_s : s \in T\}$ be a partition of V in which each set X_s contains exactly one terminal, namely, s (such a partition is called a T-*partition*); then the metric m on V, defined by $m(xy) = \mu(st)$ for all $s, t \in T$, $x \in X_s$ and $y \in X_t$, is a 0-extension. Clearly, when μ is positive, every 0-extension of μ to V one-to-one corresponds to a T-partition of V.

We consider the *minimum* 0-*extension problem* in which one is given non-negative integers (edge *widths*) $c(e) \in \mathbb{Z}_+$ for $e \in E_V$, and is required to

$$\text{Find a 0-extension } m \text{ of } \mu \text{ to } V \text{ that minimizes the value (\textit{volume}) } c \cdot m, \tag{1.1}$$

denoting by $a \cdot b$ the inner product $\sum(a(e)b(e) : e \in Q)$ of vectors $a, b \in \mathbb{R}^Q$. One can associate with c the multigraph $G_c = (V, E)$ where nodes x and y are connected by $c(xy)$ parallel edges. Then (1.1) turns into the problem of finding a T-partition $\{X_s : s \in T\}$ of V that makes the total weight of edges of G as small as possible if the weight of an edge between X_s and X_t is defined to be $\mu(st)$. Equivalently, one wishes to find a mapping $\gamma : V \to T$ that is identical on T ($\gamma(s) = s$ for $s \in T$) and minimizes the total μ-length of paths on T arising from the edges of G_c, assuming that each edge xy of G_c is transformed into a μ-shortest path between $\gamma(x)$ and $\gamma(y)$. In the latter statement, which looks closer at practical applications, the problem was called the minimum-distance mapping problem in the original version [12] of this paper.

The main result we are going to present (Theorem 1.1) concerns a relationship between the minimum 0-extension problem and its linear programming relaxation:

$$\text{Find an extension } m \text{ of } \mu \text{ to } V \text{ that minimizes } c \cdot m. \tag{1.2}$$

Let $\tau = \tau(\mu, c)$ and $\tau^* = \tau^*(\mu, c)$ denote the minima of $c \cdot m$ in (1.1) and (1.2), respectively. Since a 0-extension is an extension, $\tau \geq \tau^*$. We call μ *minimizable* if $\tau(\mu, c) = \tau^*(\mu, c)$ holds for all $V \supseteq T$ and $c : E_V \to \mathbb{Z}_+$ (this notion was introduced in [14]).

In this paper μ is usually specified as the distance function d^H of a simple connected graph $H = (T, U)$. For brevity, we speak of extensions (0-extensions) of H rather than of d^H and write $\mathcal{P}_{V,H}$, $\tau(H, c)$ and $\tau^*(H, c)$ instead of $\mathcal{P}_{V,\mathrm{d}^H}$, $\tau(\mathrm{d}^H, c)$ and $\tau^*(\mathrm{d}^H, c)$, respectively. We call H *minimizable* if d^H is minimizable. For an illustration, consider several examples.

EXAMPLE 1. Let $H = K_p$, the complete graph with p nodes. Then (1.1) is equivalent to the *minimum multiterminal* (or *multiway) cut problem* in which one is given a multigraph $G = (V, E)$ and a set $T \subseteq V$ of terminals and is asked for the removal of a minimal cardinality subset of edges of G which disconnects each terminal from all the others. This problem is NP-hard for all fixed $p \geq 3$ even if G is a simple graph [4]. On the other hand, for $p = 2$ the problem becomes the classic *minimum cut problem* for which plenty of polynomial algorithms have been designed. Moreover, in the latter case $\tau = \tau^*$ always holds, reflecting the fact that the minimum cut problem can be expressed as an integer linear program with a totally unimodular matrix (cf. [6,7]); hence, K_2 is minimizable. K_p becomes nonminimizable if $p \geq 3$. For example, if H is the complete graph on $T = \{s, t, u\}$ and the width function matches the graph drawn in Figure 1, one has $\tau = 2$ and $\tau^* = 3/2$.

EXAMPLE 2. Let $H = K_{p,r}$, the complete bipartite graph whose parts consist of p and r nodes. One can show that H is not minimizable if $p, r \geq 3$. But H is minimizable if $p \leq 2$ (cf. [10], Lemma 2.8). A 0-extension of $K_{2,r}$ is referred as a $(2, r)$-*metric*.

(A result due to Lovász [17] and Cherkassky [3] implies that for $H = K_p$, problem (1.2) has a *half-integer* optimal solution m. A similar property for $H = K_{p,r}$ follows from a result in [11, Section 3].)

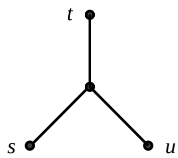

FIGURE 1.

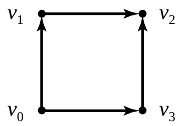

FIGURE 2.

EXAMPLE 3. Given a graph $\Gamma = (B, W)$ without triangles, let $H = (T, U)$ be obtained by partitioning each edge $e = xy \in W$ into two edges xz_e and $z_e y$ in series and then by adding a new node v and edges vz_e for all $e \in W$. The minimizability of such an H can be derived from a result in [11, Section 3].

EXAMPLE 4. Let H be the union of two graphs H' and H'' whose intersection consists of a single node. If both H' and H'' are minimizable, then H is minimizable too [14]. In particular, every tree is minimizable (as K_2 is minimizable).

In this paper we give a complete characterization of the set of minimizable graphs. To state this, we need some definitions. A *k-circuit*, C_k, is a (simple) circuit on k nodes; depending on the context, a circuit is considered as a graph (or subgraph) or as a sequence of nodes and edges in a graph.

DEFINITION 1. A subgraph $H' = (T', U')$ of H is called *isometric* if the distances in H between the nodes of H' are the same as those in H' taken separately, i.e. $\mathrm{d}^{H'}(st) = \mathrm{d}^{H}(st)$ for all $s, t \in T'$.

DEFINITION 2. H is called *orientable* if the edges of H can be oriented so that for any 4-circuit $C = (v_0, e_1, v_1, \ldots, e_4, v_4 = v_0)$, the orientations of the edges e_1 and e_3 are opposite along the circuit, and similarly for e_2 and e_4 (if e_i is directed from v_{i-1} to v_i, say, then e_{i+2} is directed from v_{i+2} to v_{i+1}); see Figure 2.

THEOREM 1.1. *The following statements are equivalent:*

(i) H is minimizable;
(ii) H is bipartite, orientable and contains no isometric k-circuit with $k \geq 6$.

Thus, the set of minimizable graphs is rather large. In particular, it includes every simple planar graph in which all inner faces are quadrangles and each node has degree not equal to three unless it is contained in the unbounded face. A special case of the latter graphs is a *grid* $\Gamma_{p,r}$ whose nodes correspond to the vectors (i, j) for $i = 0, 1, \ldots, p$ and $j = 0, 1, \ldots, r$ and edges correspond to the pairs $\{(i, j), (i', j')\}$ with $|i - i'| + |j - j'| = 1$. One can also observe from Theorem 1.1 that if H and H' are two minimizable graphs with the intersection consisting of a single edge, then $H \cup H'$ is minimizable as well.

Some interesting aspects of bipartite graphs without isometric circuits of size six or more have been studied in literature. Bandelt [2] showed that these graphs are exactly hereditary modular graphs, where a graph $H = (T, U)$ is *modular* if

$$\text{for any three nodes } s_0, s_1, s_2 \in T, \text{ there is a node } v \in T \text{ such that } \mathrm{d}^H(s_i v) + \mathrm{d}^H(v s_j) = \mathrm{d}^H(s_i s_j) \text{ for all } 0 \le i < j \le 2, \tag{1.3}$$

and *hereditary modular* if every isometric subgraph of H is modular (in the original version [12], not having known this result, we proved that H is minimizable if and only if it satisfies (1.3), is orientable, and has no isometric k-circuit with $k \ge 6$). In light of this, the graphs as in (ii) of Theorem 1.1 are orientable and hereditary modular. We call these graphs *frames*; this term is justified by the topological construction in Section 4. Thus, Theorem 1.1 asserts that the minimizable graphs are frames and only frames.

Note that the orientability of a graph can easily be recognized in polynomial time. Bandelt [2] found a polynomial algorithm to decide whether a graph is hereditary modular (alternatively, this task is simply reduced to verification of (1.3) and examination of all six-element sets of nodes because, as shown in [2], a graph that is modular but not hereditary modular must contain an isometric 6-circuit). Thus, the set of frames admits a 'good characterization'.

Concerning computational aspects of the minimum 0-extension problem, we first observe that the minimizability provides that (1.1) is as easy as (1.2).

STATEMENT 1.2. Let μ be minimizable. Then (1.1) is solvable in strongly polynomial time.

PROOF. The problem reduces to at most $|V - T||T|$ applications of (1.2). Indeed, first of all compute $a = \tau^*(\mu, c)$ (then $\tau(\mu, c) = a$ as μ is minimizable). At each iteration, choose $x \in V - T$ and identify x with some terminal s. Formally, put $V' = V - \{x\}$ and define the function c' on $E_{V'}$ by $c'(sy) = c(sy) + c(xy)$ for $y \in V - \{s, x\}$, and $c'(yz) = c(yz)$ for $y, z \in V - \{s, x\}$. Compute $a' = \tau^*(\mu, c')$. If $a' = a$, update $V := V'$ and $c := c'$ and iterate. Otherwise ($a' > a$), choose another terminal to be identified with x. Since μ is minimizable, at least one choice of a terminal for the given x does not change τ^*. As soon as the set V becomes T, an optimal 0-extension for the initial problem is constructed in an obvious way.

Since (1.2) is a linear problem whose constraint matrix consists of $O(|V|^3)$ rows and $O(|V|^2)$ columns and has all entries only 0,1 and -1, it is solvable in strongly polynomial time by use of a version of the ellipsoid method from [19]. $\square$

On the negative side, for graph metrics the problem becomes intractable for many nonminimizable graphs. Generalizing the construction elaborated in [4] for proving the NP-hardness of the minimum 3-terminal cut problem, we obtain the following result.

THEOREM 1.3. *The minimum 0-extension problem is NP-hard for all fixed non-modular graphs and all fixed non-orientable graphs, even if all widths $c(e)$ are in* $\{0, 1\}$.

Surprisingly, there are nonminimizable graphs for which (1.1) is still polynomially solvable. A class of such graphs, including the 1-skeleton graphs of hypercubes, is demonstrated in Sections 5 and 6.

We return to Theorem 1.1 and outline some ideas of our method of proof. In the background, certain polyhedral aspects of minimizable metrics are involved. For $f, g : X \to \mathbb{R}$, f is said to *dominate* g if $f(e) \ge g(e)$ for all $e \in X$. An extension m of μ to V is called *tight* if it dominates no extension except m. If m dominates no convex combination $\lambda m' + (1-\lambda)m''$, where $m', m'' \in \mathcal{P}_{V,\mu}$, $m' \ne m$, and $0 < \lambda \le 1$, then m is called an *extreme* extension of μ, or a μ-*extreme* metric. Every extreme extension is a vertex of the polyhedron $\mathcal{P}_{V,\mu}$ (but the reverse is not necessary), and because of the non-negativity of c, problem (1.2) always has an optimal solution m which is an extreme extension. Moreover, every extreme extension m is unavoidable in the sense that there exists $c : E_V \to \mathbb{Z}_+$ such that m is a unique optimal solution to (1.2) for this c. Therefore, μ is minimizable if and only if for all $V \supseteq T$, every extreme extension of μ to V is a 0-extension. This reduces our task to a proper study of extreme extensions of graph metrics. In order to recognize or construct one or another extreme metric, we rely on the property that the set of shortest paths between terminals for such a metric is inclusion maximal.

Based on these observations, we prove implication (i) $\Rightarrow$ (ii) in Theorem 1.1 by showing that if H violates at least one condition in (ii), then one can construct an extreme extension of H to some $V \supseteq T$ which is not a 0-extension. The proof of implication (ii) $\Rightarrow$ (i) is more involved. The core of it is the construction of a certain metric space over a frame H. It represents a 2-dimensional complex ('pseudo-surface') S obtained by gluing discs to the 4-circuits of H and endowed with a metric σ of ℓ_1 type, and it is proved to possess the property that every tight extension (V, m) of H is isometrically embeddable in (S, σ) (with the identity on T). For example, if H is a grid $\Gamma_{p,r}$, then S is the rectangle formed by the vectors $(\xi, \eta) \in \mathbb{R}^2$ for $0 \leq \xi \leq p$ and $0 \leq \eta \leq r$. The proof is completed by showing that any finite set W of points on S which includes T can be moved onto T so that the lengths of all σ-shortest paths on W with both ends in T are preserved; this means that T is the only set on S that induces a positive extreme extension of H.

REMARK 1.4. In the framework of a general theory of metric extensions, Isbell [8] and Dress [5] revealed the existence of the *universal tight extension* $(\mathcal{X}, \delta)$ for an arbitrary metric space (X, d). The set $\mathcal{X}$ is described as being the set of all pointwise minimal functions $f : X \to \mathbb{R}$ satisfying $f(x) + f(y) \geq d(xy)$ for all $x, y \in X$, the distance $\delta(f, g)$ of two such functions f, g is defined to be $\sup\{|f(x) - g(x)| : x \in X\}$, and each element $x \in X$ is identified with the function e_x, defined by $e_x(y) = d(xy)$ for $y \in X$. One shows that any tight extension (Y, d') of (X, d) is isometrically embeddable in $(\mathcal{X}, \delta)$. The above-mentioned space (S, σ) for a minimizable graph H is just the universal tight extension for (T, d^H); thus, we give an explicit combinatorial construction of this space in our case and reveal some additional properties which help us to prove Theorem 1.1.

This paper is organized as follows. The proof of Theorem 1.1 is given throughout Sections 2–4. Section 2 describes some properties of extreme metrics. Parts (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (i) of the theorem are proved in Sections 3 and 4, respectively. Section 5 discusses a relationship between (1.1) and the so-called multiflow locking problem and explains that the minimizability of each frame not containing $K_{2,3}$ as a subgraph can also be derived from the multiflow locking theorem in [13]. As a consequence, for every $K_{2,3}$-free frame, problem (1.1) can be solved in strongly polynomial time by a simple 'purely combinatorial' algorithm; this is generalized to some nonminimizable graphs. The concluding Section 6 proves Theorem 1.3 and raises open questions.

In conclusion of this section we demonstrate one application of minimizable graphs. In [4] one shows that the minimum k-terminal cut problem (see Example 1) can be solved by a polynomial time algorithm with the relative accuracy $2-2/k$, and is questioned for a polynomial time algorithm that provides a better guarantee for this problem. One can suggest a slightly better heuristic which relies on $(2, r)$-metric minimizations (see Example 2). More precisely, consider T, $V \supseteq T$ and $c : E_V \to \mathbb{Z}_+$, where $|T| = k$. For a T-partition $\Pi = \{X_s : s \in T\}$ of V and a subset $Z \subset T$, let $\alpha(\Pi)$ denote the sum of weights $c(xy)$ over all pairs $xy \in E_V$ connecting different sets X_s and X_t, and let $\beta(\Pi, Z)$ denote the sum of $c(xy)$'s over all pairs xy connecting X_s and X_t for $s \in Z$ and $t \in T - Z$. Then the minimum value $\alpha(\Pi)$ among the T-partitions Π is just $\tau(K_T, c)$ for the complete graph K_T on T (i.e. the minimum cardinality of a k-terminal cut in the multigraph G_c), while for a two-element subset Z of T, the minimum value $2\alpha(\Pi)-\beta(\Pi, Z)$ among all Π's is $\tau(K_{Z,T-Z}, c)$ for the complete bipartite graph $K_{Z,T-Z}$ with the parts Z and $T - Z$.

Our approximation algorithm scans all the two-element sets Z in T, for each Z solves the corresponding minimum $(2, k-2)$-metric problem, and takes the least-volume solution among these; i.e. it finds a T-partition Π and a set Z with $|Z| = 2$ for which $2\alpha(\Pi) - \beta(\Pi, Z)$ is minimum. By Statement 1.2, this is done in polynomial time (a faster algorithm solves linear problem (1.2) only once per each Z; cf. [10]). We assert that the obtained Π satisfies $\alpha(\Pi) \leq (2-q)\alpha(\Pi^*)$, where $q = 4(k-2)/k(k-1)$ and $\Pi^* = \{X_s^* : s \in T\}$ is an optimal T-partition in the minimum k-terminal cut problem. To see this, consider $Z \subset T$ such that $|Z| = 2$ and $\beta(\Pi^*, Z)$ is maximum. Then the ratio $\beta(\Pi^*, Z)/\alpha(\Pi^*)$ is at least $2(k-2)/\binom{k}{2} = q$, whence $2\alpha(\Pi^*) - \beta(\Pi^*, Z) \leq (2-q)\alpha(\Pi^*)$. Comparing this inequality with $2\alpha(\Pi) - \beta(\Pi, Z) \leq 2\alpha(\Pi^*) - \beta(\Pi^*, Z)$ (by the choice of Π) and $\alpha(\Pi) \leq 2\alpha(\Pi) - \beta(\Pi, Z)$, we get $\alpha(\Pi) \leq (2-q)\alpha(\Pi^*)$, as required. The obtained relative accuracy $2-q$ is smaller than $2 - 2/k$ for all $k \geq 4$, and is equal to $2 - 2/k = 4/3$ for $k = 3$; asymptotically, $2-q$ is $2 - 4/k + o(1/k)$. For $k = 8$, one has $2 - q = 11/7$ (Noga Alon's heuristic quoted in [4] gives the bound 12/7).

2. Extreme Metrics

Consider a metric μ on T and the set $\mathcal{P}_{V,\mu}$ of extensions of μ to a set V. The *dominant* of $\mathcal{P}_{V,\mu}$ is the set $\mathcal{D}_{V,\mu}$ of vectors (not necessarily metrics) in $\mathbb{R}^{E_V}$ that dominate some extensions, i.e.

$$\mathcal{D}_{V,\mu} = \{x \in \mathbb{R}^{E_V} : x \geq m \text{ for some } m \in \mathcal{P}_{V,\mu}\}.$$

For $m, m' \in \mathcal{P}_{V,\mu}$, m' is said to *decompose* m if there are $m'' \in \mathcal{P}_{V,\mu}$ (possibly $m'' = m'$) and $0 < \lambda < 1$ such that $m \geq \lambda m'(1-\lambda)m''$; sometimes to emphasize that this property concerns the given metric μ, we say that m' *μ-decomposes* m. So m is extreme if and only if no metric in $\mathcal{P}_{V,\mu} - \{m\}$ decomposes m, or if and only if m is a vertex of $\mathcal{D}_{V,\mu}$. Clearly if m' decomposes m, then $m'(xy) = 0$ whenever $m(xy) = 0$. This implies that if m is a 0-extension of μ, and m' μ-decomposes m, then m' coincides with m; therefore, every 0-extension is extreme. The next statement characterizes the minimizability in terms of extreme metrics.

STATEMENT 2.1. *A metric μ on T is minimizable if and only if for all $V \supseteq T$, every μ-extreme metric on V is a 0-extension.*

PROOF. By linear programming arguments, for each $c : E_V \to \mathbb{Z}_+$, the minimum of $c \cdot m$ over $m \in \mathcal{P}_{V,\mu}$ is achieved by a vertex of $\mathcal{D}_{V,\mu}$; conversely, for each vertex m of $\mathcal{D}_{V,\mu}$, there is $c : E_V \to \mathbb{Z}_+$ such that $c \cdot m < c \cdot m'$ for any other vector m' in $\mathcal{P}_{V,\mu}$. This implies the statement. □

In this and the following sections, when proving necessary conditions for the minimizability, we will consider a certain submetric μ' of the distance function d^H of some graph H, construct a 'bad' μ'-extreme metric m' and then use this m' to show the existence of an extreme extension of H that is not a 0-extension, thus concluding that H is not minimizable. This relies on three simple facts stated in Statements 2.2–2.4. For a metric h on W, $h_{|W'}$, stands for its submetric on $W' \subseteq W$ (or the *restriction* of h to W').

STATEMENT 2.2. *Let μ be a metric on T, and μ' its restriction to $T' \subseteq T$. Let m' be a μ'-extreme metric on a set V' with $V' \cap T = T'$. Then m' can be extended to a μ-extreme metric m on $V = V' \cup T$.*

PROOF. Define the function h by $h(xy) = m'(xy)$ for $x, y \in V'$, $h(xy) = \mu(xy)$ for $x, y \in T$, and $h(xy) = \min\{m'(xs) + \mu(sy) : s \in T'\}$ for $x \in V'$ and $y \in T - T'$. One can check that h is a metric on V. Moreover, $h \in \mathcal{P}_{V,\mu}$ (since $h_{|T} = \mu$). Take a μ-extreme metric m on V that μ-decomposes h. Then $m_{|V'}$ μ'-decomposes m'. The fact that m' is μ'-extreme implies $m_{|V'} = m'$. Therefore, m is the desired metric. □

Statements 2.1 and 2.2 show that

$$\text{if } \mu \text{ is a minimizable metric on } T,\ \mu' \text{ is its restriction to } T' \subseteq T, \text{ and } m' \text{ is a } \mu'\text{-extreme metric on a set } V' \supseteq T' \text{ with } V \cap T = T', \text{ then there is a mapping } \gamma : V' \to T \text{ such that } \gamma \text{ is identical on } T' \text{ and satisfies } m'(xy) = \mu(\gamma(x)\gamma(y)) \text{ for all } x, y \in V'. \tag{2.1}$$

STATEMENT 2.3. *Let μ be a minimizable integer metric on T. Then: (i) for any sequence $Q = (s_0, \dots, s_k)$ of elements of T, the number $\mu(Q) = \mu(s_0s_1) + \dots + \mu(s_{k-1}s_k) + \mu(s_ks_0)$ is even; and (ii) for any $s_0, s_1, s_2 \in T$, there is $v \in T$ such that $\mu(s_iv) + \mu(vs_j) = \mu(s_is_j)$ for all $0 \le i < j \le 2$.*

PROOF. Observe that (ii) implies (i). Indeed, if (i) is false, choose a minimum cardinality sequence $Q = (s_0, \dots, s_k)$ for which $\mu(Q)$ is odd. Then $k = 2$ (for if $k \ge 3$, then both $Q' = (s_0, s_1, s_2)$ and $Q'' = (s_0, s_2, \dots, s_k)$ are smaller sequences, and one of $\mu(Q')$ and $\mu(Q'')$ is, obviously, odd). But the v as in (ii) gives $\mu(Q) = 2\sum(\mu(s_iv) : i = 0, 1, 2)$. Hence, $\mu(Q)$ is even; a contradiction.

To see (ii), assume that s_0, s_1, s_2 are different (otherwise the statement is trivial), and let numbers p_0, p_1, p_2 be such that $p_i + p_j = \mu(s_is_j)$ for $0 \le i < j \le 2$. These numbers are unique (namely, $p_i = \frac{1}{2}(\mu(s_is_j) + \mu(s_is_k) - \mu(s_js_k))$, where $\{i, j, k\} = \{0, 1, 2\}$) and non-negative (since μ is a metric). This implies that the submetric μ' of μ on the set $T' = \{s_0, s_1, s_2\}$ is extended to the metric m' on the four-element set $V' = T' \cup \{x\}$ satisfying $m'(s_ix) = p_i$ for $i = 0, 1, 2$; moreover, this m' is μ'-extreme. Take a mapping γ as in (2.1). Then $v = \gamma(x)$ is as required. □

By (i) in Statement 2.3, any minimizable graph is bipartite. Moreover, (ii) in this statement gives a sharper property, namely: any minimizable graph H satisfies (1.3), i.e. H is modular. We will use the following corollary from (2.1) and Statement 2.3:

$$\text{if } \mu \text{ is an integer minimizable metric, then for any submetric } \mu' \text{ of } \mu \text{ and any extreme extension } m' \text{ of } \mu' \text{ to a set } V', m' \text{ is integral and } m'(x_0x_1) + \dots + m'(x_{k-1}x_k) + m'(x_kx_0) \text{ is even for any } x_0, \dots, x_k \in V'. \tag{2.2}$$

The next statement, which has the same flavor as Statement 2.2, shows that we can construct extreme metrics recursively.

STATEMENT 2.4. *Let $\mu_1, \dots, \mu_k$ be metrics on $T_1, \dots, T_k$, respectively, and let $m_1, \dots, m_k$ be metrics on $V_1, \dots, V_k$, respectively, where $k > 1$ and: (i) for $i = 2, \dots, k$, $V_i \cap (V_1 \cup \dots \cup V_{i-1}) = T_i \subseteq V_{i-1}$; (ii) each m_i is an extreme extension of μ_i; and (iii) each μ_i is a submetric of m_{i-1}, for $i > 1$. Then there exists a μ_1-extreme metric $\tilde{m}$ on $\tilde{V} = V_1 \cup \dots \cup V_k$ that coincides with each m_i within V_i.*

PROOF. By induction on k, assume that there exists a μ_2-extreme metric g on the set $W = V_2 \cup \dots \cup V_k$ such that for each $i = 2, \dots, k$, g coincides with m_i within V_i (this is trivial for $k = 2$). Condition (i) implies that $W \cap V_1 = T_2$. Apply Statement 2.2 to $T = V_1$, $\mu = m_1$, $T' = T_2$, $\mu' = \mu_2$ and $m' = g$. This gives a m_1-extreme metric $\tilde{m}$ on $\tilde{V} = V \cup W$ such that $\tilde{m}_{|W} = g$. We have $\tilde{m}_{|V_i} = m_i$ for $i = 1, \dots, k$. Clearly $\tilde{m}$ is an extension of μ_1. We assert that $\tilde{m}$ is μ_1-extreme. To see this, consider the expression $\tilde{m} \ge \lambda m' + (1 - \lambda)m''$, where m', m'' are extensions of μ_1 to $\tilde{V}$ and $0 < \lambda < 1$. Since $\tilde{m}_{|V_1} = m_1$, and m_1 is μ_1-extreme, both m' and m'' coincide with m_1 within V_1. Hence, both m' and m'' are extensions of m_1. Now the fact that $\tilde{m}$ is m_1-extreme implies $m' = m'' = \tilde{m}$. Thus, $\tilde{m}$ is the desired μ_1-extreme metric. □

Next we discuss one approach to proving that some metrics are extreme. We need some terminology.

A sequence $P = (x_0, x_1, \dots, x_k)$ of elements of V is called an $x_0 - x_k$ *path* on V; for brevity we use notation $x_0x_1 \dots x_k$ for P. If $x_{i-1} \ne x_i$, the pair $x_{i-1}x_i$ is called an *edge* of P; $|P|$ denotes the number of edges of P. If each edge of P is an edge of a graph $G = (V, E)$, P is a path of G. If all x_i's are different, P is *simple*. If x_0 and x_k are distinct terminals (in T), P is called a T-*path*. Given $m \in \mathcal{P}_{V,\mu}$, a path $P = x_0x_1 \dots x_k$ is called a μ-*geodesic* of m if P is a T-path which is shortest for m, i.e. the m-*length* $m(P) = m(x_0x_1) + \dots + m(x_{k-1}x_k)$

of P equals $m(x_0x_k)(= \mu(x_0x_k))$. The set of μ-geodesics of m is denoted by $\mathcal{G}(m) = \mathcal{G}_\mu(m)$. The following fact is easy (see, e.g., [15]):

let $m, m' \in \mathcal{P}_{V,\mu}$ and let m be positive; then m' decomposes m if and only if $\mathcal{G}(m) \subseteq \mathcal{G}(m')$; in particular, every positive μ-extreme metric is uniquely determined by the set of its μ-geodesics. (2.3)

Borrowing some ideas from [1, 15], we give a practical application of property (2.3) for constructing certain extreme metrics (such a construction will be crucial in the next section). Let $G = (V, E)$ be a connected graph whose distance function is proportional to some extension of μ, i.e. $\alpha \mathrm{d}^G \in \mathcal{P}_{V,\mu}$ for some $\alpha \in \mathbb{R}_+$. A subgraph $G' = (V', E')$ of G is called *μ-isometric* if it is isometric and

for each $x, y \in V'$, there are $s, t \in T$ such that $\mathrm{d}^G(sx) + \mathrm{d}^G(xy) + \mathrm{d}^G(yt) = \mu(st)/\alpha$; (2.4)

in other words, the metric $\alpha\mathrm{d}^G$ has a μ-geodesic which includes, as a part, a path in G' between x and y.

Suppose that G as above has a μ-isometric even circuit $C = x_0x_1 \ldots x_{2k}(x_{2k} = x_0)$. For $G' = C$, (2.4) is equivalent to the property that

for $i = 0, \ldots, k-1$, there are $s_i, s_{i+k} \in T$ such that $\mathrm{d}^G(s_ix_i) + \mathrm{d}^G(x_{i+k}s_{i+k}) + k = \mu(s_is_{i+k})/\alpha$. (2.5)

Let m' μ-decompose $\alpha\mathrm{d}^G$. Then $\mathcal{G}(m) \subseteq \mathcal{G}(m')$ (by (2.3)). Therefore, (2.5) implies that

$$m'(s_ix_i) + m'(x_{i+k}s_{i+k}) + \sum_{j=i}^{i+k-1} m'(x_jx_{j+1}) = \mu(s_is_{i+k}), \qquad i = 0, \ldots, 2k-1; \tag{2.6}$$

taking indices modulo $2k$. Putting together the $2k$ equalities in (2.6), we obtain

$$m'(x_ix_{i+1}) = m'(x_{i+k}x_{i+k+1}), \qquad i = 0, \ldots, k-1. \tag{2.7}$$

Let us call edges $e, e' \in E$ *μ-dependent* if there is a sequence $e = e_0, e_1, \ldots, e_q = e'$ of edges and a sequence $C_1, \ldots, C_q$ of μ-isometric even circuits of G such that e_{j-1} and e_j are opposite edges of C_j, for $j = 1, \ldots, q$. A maximal set of μ-dependent edges of G is called an *orbit* (or *μ-orbit*). We say that G is *μ-dependent* if each orbit of G includes the edge set of some μ-geodesic (e.g. includes an edge with both ends in T). For example, G is μ-dependent if it has only one orbit. The following statement is a slight generalization of one result on extreme graph metrics in [1, 15] (see also [16]).

STATEMENT 2.5. *Let $G = (V, E)$ be a connected graph with $V \supseteq T$ such that $m = \alpha\mathrm{d}^G \in \mathcal{P}_{V,\mu}$ for some $\alpha \in \mathbb{R}_+$. (i) If G is μ-dependent and m' is a μ-extreme metric on V that μ-decomposes m, then m' and m coincide on E. (ii) If G is μ-dependent and μ-isometric, then m is μ-extreme.*

PROOF. To see (i), observe from (2.7) that $m'(e)$ is the same number b for all edges e of an orbit Q. Since Q includes the edge set of a μ-geodesic P, b is exactly $\mu(st)/|P|$, where s, t are the ends of P. Hence, m' is uniquely determined on each edge e of G, implying $m'(e) = m(e)$. Part (ii) follows from (i) and the fact that any two $x, y \in V$ belong to a μ-geodesic with all edges in G (whence the distance between x and y is uniquely determined). □

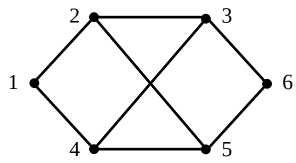

FIGURE 3. $K_{3,3}^-$

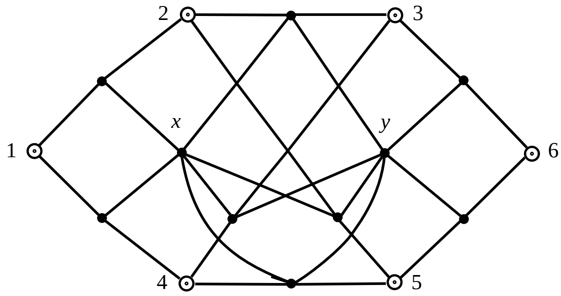

FIGURE 4.

3. NONMINIMIZABLE GRAPHS

In the previous section we showed that a minimizable graph is bipartite. In this section we show that a minimizable graph is orientable and has no isometric k-circuit with $k \geq 6$, thus proving part (i) $\Rightarrow$ (ii) in Theorem 1.1. We use prefix H in place of d^H in terms H-extreme, H-geodesic, H-isometric, etc. Notation $A \simeq B$ stands for isomorphic graphs A and B. In our proof the graph $K_{3,3}^-$ that is obtained from $K_{3,3}$ by deleting one edge (see Figure 3) plays an important role. Note that $K_{3,3}^-$ is nonorientable.

LEMMA 3.1. *For $H' = (T', U') \simeq K_{3,3}^-$, there is an H'-extreme extension which is not integral.*

PROOF. Let the nodes of H' be numbered by $1, \dots, 6$ as indicated in Figure 3. Split each edge $e = ij$ of H' into two edges iz_e and $z_e j$ in series, add two extra nodes x and y, add edges xz_e for all $e = ij \in U'$ with $i, j \leq 5$, and add edges yz_e for all $e = ij \in U'$ with $i, j \geq 2$. The resulting graph, denoted by $G = (V, E)$, is shown in Figure 4.

A routine, though somewhat tiresome, check (which is facilitated by using symmetries of G) shows that: (i) $\frac{1}{2}\mathrm{d}^G(ij) = \mathrm{d}^H(ij)$ for all $i, j \in T'$, i.e. $m = \mathrm{d}^G/2$ is an extension of H'; (ii) G is H'-isometric (whence each 4-circuit of G is H'-isometric since G is bipartite); and (iii) any two edges of G are H'-dependent, i.e. G has only one orbit. Also m is not integral on the edges of G. Now the result follows from (2.2) and Statement 2.5. □

Note that any induced subgraph $K_{3,3}^-$ of a bipartite graph is always isometric. A feature of $K_{3,3}^-$ is demonstrated by the following statement.

STATEMENT 3.2. *Let $H' = (T', U')$ be an isometric subgraph of a graph H. Suppose that there is a connected graph $G = (V, E)$ with $V \supseteq T'$ such that: (i) $\alpha\mathrm{d}^G$ is an extension of H' for some $0 < \alpha \leq 1$; (ii) G is H'-dependent; and (iii) G contains an H'-isometric subgraph $H'' = (T'', U'') \simeq K_{3,3}^-$. Then H is not minimizable. In particular, H is not minimizable if it contains $K_{3,3}^-$ as an isometric subgraph.*

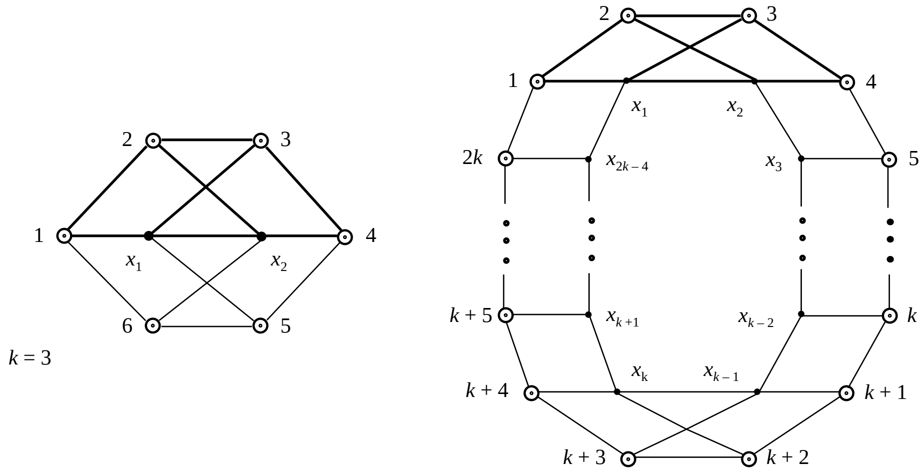

FIGURE 5.

PROOF. Let m_1 be an H'-extreme metric on V that H'-decomposes αd^G. By Statement 2.5, $m_1(e) = \alpha$ for all $e \in E_1$. Moreover, since H'' is H'-isometric and every H'-geodesic of αd^G is an H'-geodesic of m_1, the restriction of m_1 to T'' is $\mu_2 = \alpha d^{H''}$. Take a noninteger extreme extension h of H'' to a set W (existing by Lemma 3.1). Then $m_2 = \alpha h$ is μ_2-extreme and not integral (as $\alpha \leq 1$). Assuming that $W \cap V = T''$, apply Statement 2.4 to these $\mu_1 = d^{H'}$, m_1, μ_2, m_2 to obtain an H'-extreme metric m' on $W \cup V$ that coincides with m_2 on W. Finally, transform m' to an H-extreme metric m according to Statement 2.2. Then m is not integral, so H is not minimizable. □

This statement enables us to eliminate graphs with big isometric circuits.

LEMMA 3.3. *Let H contain an isometric $2k$-circuit C with $k \geq 3$. Then H is not minimizable.*

PROOF. Let the nodes of this circuit $C = (T', U')$ be numbered by $1, \dots, 2k$ in this order. The result follows from the existence of a graph $G = (V, E)$ which satisfies properties (i)–(iii) in the hypotheses of Statement 3.2 (with $H' = C$). The desired graphs G (for $k = 3$ and $k \geq 4$) are depicted in Figure 5; here $\alpha = 1$ and the corresponding subgraph $H'' \simeq K^-_{3,3}$ is drawn in bold.

One can check that: (a) in both cases, $d^G{}_{|T'} = d^C$, and H'' is a C-isometric subgraph; (b) for $k = 3$, G has only one C-orbit; (c) for $k \geq 4$, there are $k - 2$ orbits $O_1, \dots, O_{k-2}$, each containing an edge of C; namely, for $i = 2, \dots, k-2$, O_i is formed by the edges $\{i+2, i+3\}, \{x_i, x_{i+1}\}, \{k+i+2, k+i+3\}, \{x_{k+i-2}, x_{k+i-1}\}$ (letting $x_{2k-3} = x_1$ and identifying the terminal 1 with $2k+1$), and O_1 is the rest. Thus, G is as required. □

For further purposes we need two more 'nonminimizable configurations'. As mentioned in the Introduction, $H' = K_{p,r}$ is not minimizable for $p, r \geq 3$. This is because adding to H' a node x and edges xs for all nodes s in H' results in the graph G for which d^G is an extension of H', and one can easily show that G is H'-extreme. Since G is not bipartite, H' is not minimizable, by (2.2). Note also that any subgraph $K_{p,r}$ of a bipartite graph is isometric.

Another example is given by the graph C_6^+ depicted in Figure 6. This graph has an isometric 6-circuit C (that on the nodes $1, 2, \dots, 6$); so C_6^+ is not minimizable. Adding to C_6^+ one or two edges connecting opposite nodes in C creates an isometric subgraph $K^-_{3,3}$ in the resulting graph, while adding three such edges turns C_6^+ into $K_{3,4}$.

The above observations can be summarized as follows.

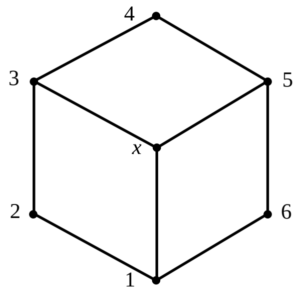

FIGURE 6. C_6^+

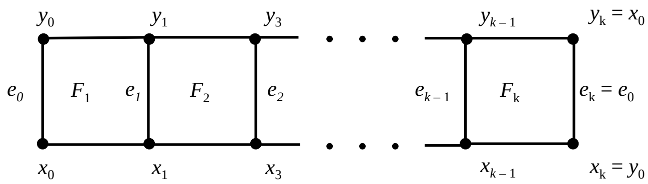

FIGURE 7. An orientation-reversing dual cycle.

STATEMENT 3.4. *If H has a subgraph isomorphic to $K_{3,3}^-$ or C_6^+, then H is not minimizable.*

REMARK 3.5. If H contains an isometric k-circuit with $k \geq 6$ or an isometric subgraph $K_{3,3}^-$, then problem (1.2) has unbounded fractionality, in the sense that for any $q \in \mathbb{Z}_+$ there is an H-extreme metric m with the denominator of some component of m at least q. This is provided by the existence of an H'-isometric subgraph $K_{3,3}^-$ in the graph G which occurred in the proof of Lemma 3.1 and is drawn in Figure 4, where H' is the graph $K_{3,3}^-$ on $\{1, \ldots, 6\}$ as in Figure 3. Statement 2.4 enables us to iterate the expansion of such subgraphs $K_{3,3}^-$ into metrics proportional to G, obtaining a sequence of H-extreme metrics with increasing denominators. In contrast, as mentioned in the Introduction, the graphs K_p and $K_{p,r}$ $(p, r \geq 3)$, though being nonminimizable, have the property that all denominators occurring in their extreme extensions are at most two.

Next we eliminate the remaining case in (ii) of Theorem 1.1 by showing that every nonorientable graph is nonminimizable. In view of (2.2), it is enough to consider only bipartite nonorientable graphs H. We apply somewhat different techniques. By a *dual path* of H we mean a sequence $D = (e_0, F_1, e_1, \ldots, F_k, e_k)$, where $e_0, \ldots, e_k$ are edges and $F_1, \ldots, F_k$ are 4-circuits in H, and e_{i-1}, e_i are opposite edges in F_i. When $e_0 = e_k$, D is a *dual cycle*.

LEMMA 3.6. *Let $H = (T, U)$ be bipartite and nonorientable. Then H is nonminimizable.*

PROOF. The nonorientability of H is equivalent to the existence of an *orientation-reversing* dual cycle $D = (e_0, F_1, e_1, \ldots, F_k, e_k)$ in H. This means that if the ends of each edge e_i are denoted by x_i and y_i so that each circuit F_i becomes $x_{i-1}y_{i-1}y_ix_ix_{i-1}$ (up to reversing and cyclically shifting F_i), then the ends of the edge $e_k = e_0$ satisfy $x_k = y_0$ and $y_k = x_0$; see Figure 7 for an illustration.

We assume that such a D is chosen so that its length $|D| = k$ is as small as possible, consider D up to shifting cyclically, let $F_0 = F_k$, and take indices modulo k. The minimality of D implies that all $e_1, \ldots, e_k$ are different (while some F_i and F_j may coincide). Observe

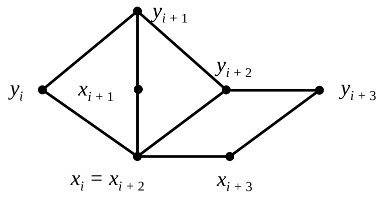

FIGURE 8.

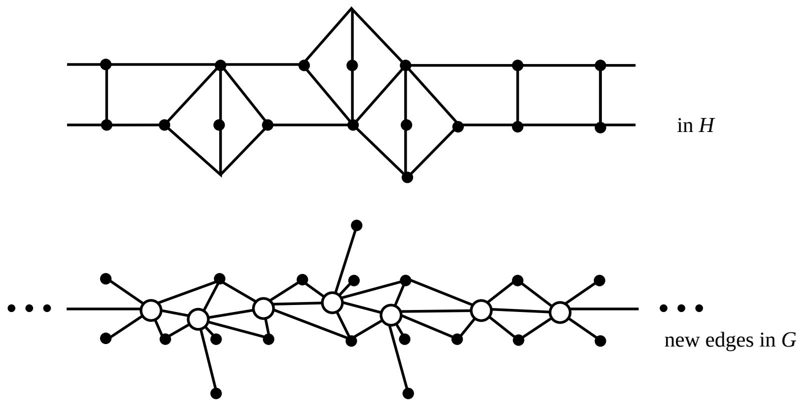

FIGURE 9.

that

$$\text{if } x_i = y_j \text{ then } |i - j| \text{ is odd, and if } x_i = x_j \text{ or } y_i = y_j \text{ then } |i - j| \text{ is even;} \tag{3.1}$$

otherwise H is not bipartite. In view of Statement 3.4, we may assume that H contains no subgraph $K^-_{3,3}$. This implies that

$$\begin{array}{l}\text{if } x_i = x_{i+2} \text{ for some } i\text{, then all other nodes among } x_i, y_i, \dots, x_{i+3}, y_{i+3} \\ \text{are different, and similarly if } y_i = y_{i+2};\end{array} \tag{3.2}$$

see Figure 8. Indeed, one can see that if this is not so, then H has a subgraph $K^-_{3,3}$ or D is nonminimal (e.g. the latter occurs when $x_i = x_{i+2}$ and $y_{i+1} = y_{i+3}$, in which case the part from F_{i+1} to F_{i+3} in D can be replaced by the 4-circuit $x_i y_i y_{i+3} x_{i+3} x_i$). We say that F_i and F_{i+1} are *squeezed* if either $x_{i-1} = x_{i+1}$ or $y_{i-1} = y_{i+1}$. By (3.2), in this case F_{i+1} and F_{i+2} are nonsqueezed, and similarly for F_{i-1} and F_i. By (3.1), if F_i and F_{i+1} are nonsqueezed, then the nodes $x_{i-1}, y_{i-1}, x_{i+1}, y_{i+1}$ are different.

Form the graph $G' = (V', E')$ by adding to H new nodes z_i and edges connecting z_i to each $x_{i-1}, y_{i-1}, x_i, y_i$ for $i = 1, \dots, k$ (note that z_i and z_j are different even if $F_i = F_j$). Transform G' into the graph $G = (V, E)$ as follows. For $i = 1, \dots, k$, if F_i and F_{i+1} are squeezed, identify z_i and z_{i+1} (and identify the multiple edges that appear); and if F_i and F_{i+1} are nonsqueezed, add the edge $u_i = z_i z_{i+1}$ (see Figure 9). We need two claims.

CLAIM 1. d^G *is an extension of* d^H.

PROOF. Suppose that this is not so. Then G has a T-path $P = v_0 v_1 \dots v_q$ whose length $|P| = q$ is less than $\mathrm{d}^H(v_0 v_q)$. Choose such a P so that $|P|$ is minimum. Then the nodes $v_1, \dots, v_{q-1}$ of P are not in T. Since no edge in $E - U$ connects terminals, $|P| > 1$. By the construction of G, if $|P| = 2$, then some i satisfies at least one of the following: (i) $v_1 = z_i$, and both v_0, v_2 are in F_i; or (ii) F_i and F_{i+1} are squeezed, $v_1 = z_i = z_{i+1}$, and both v_0, v_2 are in $F_i \cup F_{i+1}$. But in both cases, $\mathrm{d}^H(v_0v_2) \le 2$.

Thus, $|P| \ge 3$. Then the edge $e = v_1v_2$ is of the form z_iz_{i+1} for some i; assume that $v_1 = z_i$. If v_0 is x_i (or y_i), then v_0 and v_2 are connected by an edge in G; therefore, replacing the part $v_0v_1v_2$ by v_0v_2 makes a shorter v_0–v_q path, contradicting the minimality of P. If v_0 is x_{i-1} (case $v_0 = y_{i-1}$ is similar), then replacing the part $v_0v_1v_2$ by the path $v_0x_iv_2$ makes a v_0–v_q path P' with $|P'| = |P|$. But the subpath P'' of P' from x_i to v_q is a T-path, and $|P''| = |P'| - 1 < \mathrm{d}^H(v_0v_q) - 1 \le \mathrm{d}^H(x_iv_q)$, contrary to the minimality of P. Finally, if F_{i-1} and F_i are squeezed, $x_i = x_{i-2}$ say, and v_0 is in F_{i-1} but not in F_i (i.e. $v_0 = y_{i-2}$), then $v_0x_iv_2$ is again a path in G, and we get a contradiction arguing as above. □

By Claim 1, $d^H \in \mathcal{P}_{V,H}$. Choose an H-extreme metric m' on V that H-decomposes d^G. We use the property that each geodesic of d^G is a geodesic of m' too. For $i = 1, \ldots, k$, let $\alpha_i = m'(x_{i-1}z_i)$, $\alpha'_i = m'(x_iz_i)$, $\beta_i = m'(y_{i-1}z_i)$ and $\beta'_i = m'(y_iz_i)$.

CLAIM 2. *For* $i = 1, \ldots, k$, $\alpha_i - \beta_i = \alpha_{i+1} - \beta_{i+1}$.

PROOF. Consider two possible cases.

(i) F_i and F_{i+1} are not squeezed. Then $\mathrm{d}^H(x_{i-1}y_{i+1}) = \mathrm{d}^H(y_{i-1}x_{i+1}) = 3$ (for if, say, x_{i-1} and y_{i+1} are connected by an edge e in H, then adding e to $F_i \cup F_{i+1}$ forms $K^-_{3,3}$). Hence, both $P = x_{i-1}z_iz_{i+1}y_{i+1}$ and $P' = y_{i-1}z_iz_{i+1}x_{i+1}$ are H-geodesics of G. Then $m'(P) = m'(P') = 3$. This implies $\alpha_i + \beta'_{i+1} = \beta_i + \alpha'_{i+1}$. This together with the obvious equalities $\alpha_{i+1} + \beta'_{i+1} = 2$ and $\beta_{i+1} + \alpha'_{i+1} = 2$ gives $\alpha_i - \beta_i = \alpha_{i+1} - \beta_{i+1}$.

(ii) F_i and F_{i+1} are squeezed, $x_{i-1} = x_{i+1}$ say. The T-paths $y_{i-1}z_ix_i$, $x_iz_iy_{i+1}$ and $y_{i-1}z_iy_{i+1}$ are geodesics, whence $\beta_i = \alpha_{i+1} = \beta'_{i+1} = 1$. Also $x_{i-1}z_iy_i$ is a geodesic, whence $\alpha_i + \beta_{i+1} = 2$. Hence, $\alpha_i - \beta_i = (2 - \beta_{i+1}) - 1 = 1 - \beta_{i+1} = \alpha_{i+1} - \beta_{i+1}$. □

Now we finish the proof of the lemma as follows. We derive from Claim 2 that $\alpha_1 - \beta_1 = \alpha_{k+1} - \beta_{k+1}$. Also $\beta_{k+1} = \alpha_1$ and $\alpha_{k+1} = \beta_1$ since $x_0 = y_k$ and $y_0 = x_k$. Hence, $m'(x_0z_1) = \alpha_1 = \beta_1 = m'(y_0z_1)$. This implies that if α_1 is an integer, then the m'-length of the circuit $x_0y_0z_1x_0$ is odd. Thus, H is not minimizable, by (2.2). □ □

This completes the proof of part (i) ⇒ (ii) in Theorem 1.1.

4. EMBEDDING IN A PSEUDOSURFACE

In this section we show that if $H = (T, U)$ is a frame (i.e. it satisfies (ii) in Theorem 1.1), then H is embeddable in a 2-dimensional space $S = S^H$ endowed with a certain metric $\sigma = \sigma^H$ so that every tight extension (V, m) of H admits a (unique) isometric embedding in (S, σ) (Theorem 4.2). Conversely, every finite subset $V \supseteq T$ on S induces a tight extension of H. We then explain that, given a finite set $V \subset S$, each point in $V - T$ can be moved into a point in T so as to preserve the length of every σ-shortest T-path on V (Lemma 4.3). This will prove (ii) ⇒ (i) in Theorem 1.1.

Let $H = (T, U)$ be a frame (as before, H is connected and has no multiple edges). We assume that H is 2-edge-connected; this does not lead to loss of generality, by the argument in Example 4 in the Introduction. Then each edge e of H belongs to a circuit; moreover, e belongs to a 4-circuit (because H is bipartite, has no isometric k-circuit with $k > 4$, and any minimum length circuit containing e is isometric). A maximal subgraph $K_{2,r}$ of H is called a 2-*clique*. Note that any 4-circuit is contained in exactly one 2-clique (otherwise H contains $K^-_{3,3}$). The space S that we now construct is defined by the 2-cliques of H.

First, each edge uv of H is regarded as being homeomorphic to the closed interval (segment) $[0, 1] \subset \mathbb{R}^1$, or it consists of points z at distance α from u and $1-\alpha$ from v, for all $0 \le \alpha \le 1$.

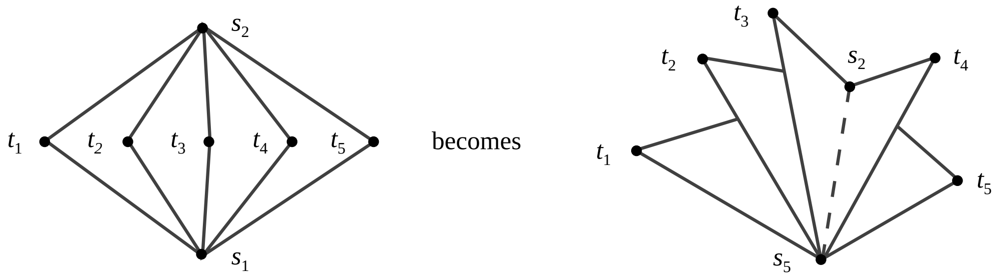

FIGURE 10. Creation of a folder.

Each 4-circuit $C = v_0v_1v_2v_3v_0$ (considered up to reversing and/or cyclically shifting) expands into a 2-dimensional disc $D = D^C$. Formally, D is homeomorphic to $[0, 1] \times [0, 1] \subset \mathbb{R}^2$, nodes v_0, v_1, v_2, v_3 are identified with points (0,0), (0,1), (1,1), (1,0), respectively, and the edges of C are identified with the corresponding straight-line segments in D; e.g., each point z in the edge v_1v_2 is identified with the point $(\xi, 1)$ in D, where ξ is the distance of z from v_1. In what follows we do not distinguish between a node (edge) of C and the corresponding point (segment) in D^C.

Second, if 4-circuits C and C' have a single node (edge) in common, the corresponding points (segments) in D^C and $D^{C'}$ are identified accordingly.

Third, suppose that 4-circuits $C = v_0v_1v_2v_3v_0$ and $C' = u_0u_1u_2u_3u_0$ have two common edges, say, $v_i = u_i$ for $i = 0, 1, 2$. We identify the corresponding halves in D^C and $D^{C'}$. Formally, assuming for definiteness that v_0, v_1, v_2 are represented as the points (0,0), (0,1), (1,1) in both D^C and $D^{C'}$, we identify each point (ξ, η) with $0 \le \xi \le \eta \le 1$ in D^C with the point (ξ, η) in $D^{C'}$. Under these operations, every 2-clique H' with node parts $\{s_1, s_2\}$ and $\{t_1, \dots, t_r\}$ produces the shape $F^{H'}$, called the *folder* for H', which is homeomorphic to the space obtained from r copies of the triangle $\{(\xi, \eta) : 0 \le \xi \le \eta \le 1\}$ by sticking them together along the 'diagonal' $\{(\alpha, \alpha) : 0 \le \alpha \le 1\}$, see Figure 10 for $r = 5$.

The resulting space is just the desired $S = S^H$. We call the region D^C for each 4-circuit C a *cell* of S (the interiors of such cells within the same folder are intersecting, in contrast to the standard notion of a cell in a cellular complex (cf., e.g., [18]); S turns into a simplicial complex by subdividing our cells into corresponding triangles). The orientability of H implies that S is orientable.

We assign an ℓ_1-metric σ^C within each cell D^C in a natural way. More precisely, the above representation of D^C establishes cartesian coordinates (ξ, η) in D^C, and the distance $\sigma^C(xy)$ between points $x = (\xi', \eta')$ and $y = (\xi'', \eta'')$ of D^C is defined to be $|\xi' - \xi''| + |\eta' - \eta''|$. We extend these metrics to the global metric $\sigma = \sigma^H$ on S in a natural way, by defining $\sigma(xy)$, $x, y \in S$, to be the infimum of values $\hat{\sigma}(\mathcal{P}) = \sigma^{C_1}(x_0x_1) + \cdots + \sigma^{C_k}(x_{k-1}x_k)$ among all finite sequences $\mathcal{P} = (x_0, x_1, \dots, x_k)$ of points on S such that $x_0 = x$, $x_k = y$, and each pair x_{i-1}, x_i belongs to the same cell, namely, D^{C_i} (observe that $\sigma^{C_i}(x_{i-1}x_i)$ does not depend on the choice of a cell C_i containing x_{i-1} and x_i). The fitness of σ is shown in the following statement.

STATEMENT 4.1.

(i) $\sigma^H(st) = \mathrm{d}^H(st)$ *for all* $s, t \in T$.
(ii) For each 4-circuit C, σ^H *coincides with* σ^C *within* D^C.

PROOF. Consider $\mathcal{P} = (x_0, \dots, x_k)$ as above and observe that if $C_i, \dots, C_j$ belong to the same 2-clique Q, then $x_{i-1}, x_j \in D^C$ for some 4-circuit C in Q, and $\sigma^C(x_{i-1}x_j) \le$

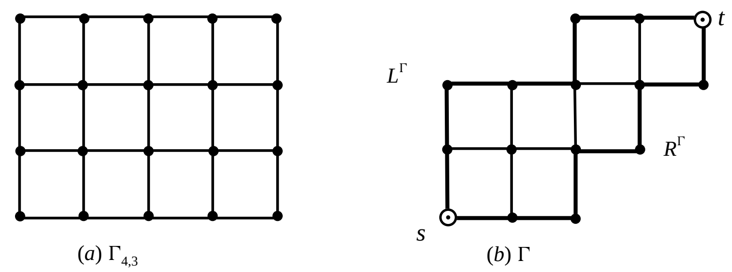

FIGURE 11. A grid and a net.

$\sigma^{C_i}(x_{i-1}x_i) + \cdots + \sigma^{C_j}(x_{j-1}x_j)$. Therefore, removing $x_i, \ldots, x_{j-1}$ from $\mathcal{P}$ results in the sequence $\mathcal{P}'$ with $\hat{\sigma}(\mathcal{P}') \le \hat{\sigma}(\mathcal{P})$. Next, if for some i, the points x_{i-1}, x_i lie on different edges e, e' in C_i which share a common node v, then inserting v between x_{i-1} and x_i results in $\mathcal{P}'$ with $\hat{\sigma}(\mathcal{P}') = \hat{\sigma}(\mathcal{P})$.

Let $x_0 = s \in T$ and $x_k = t \in T$. Using the above observations, one can transform $\mathcal{P}$ into a (feasible) sequence $\mathcal{P}'$ with the beginning s and end t so that $\hat{\sigma}(\mathcal{P}') \le \hat{\sigma}(\mathcal{P})$ and all elements of $\mathcal{P}'$ are in T. Then $\hat{\sigma}(\mathcal{P}') \ge \mathrm{d}^H(st)$, yielding (i). To see (ii), let both $x_0 = x$ and $x_k = y$ be in some D^C. Using the above observations, one can transform $\mathcal{P}$ into $\mathcal{P}' = (y_0, y_1, \ldots, y_q)$ such that $y_0 = x$, $y_q = y$, $\hat{\sigma}(\mathcal{P}') \le \hat{\sigma}(\mathcal{P})$, and either (a) $q = 1$, or (b) $q \ge 4$, both y_1 and y_{q-1} lie on the boundary of the folder including D_C, and for $i = 2, \ldots, q-1$, the points y_{i-1} and y_i lie on the opposite edges of a 4-circuit C_i (allowing $y_0 = y_1$ and $y_{q-1} = y_q$). In case (a), $\hat{\sigma}(\mathcal{P}') = \sigma^C(xy)$. In case (b), $\sigma^{C_i}(y_{i-1}y_i) \ge 1$ for $i = 2, \ldots, q-1$, whence $\sigma(\mathcal{P}') \ge 2 \ge \sigma^C(xy)$. This gives (ii). □

A simple example of a frame is a grid $\Gamma_{p,r}$, or a $p \times r$ *grid*, defined in the Introduction (see Figure 11(a) where $p = 4$, $r = 3$). Here each 2-clique is a 4-circuit, S forms the rectangle $\{(\xi, \eta) : 0 \le \xi \le p, 0 \le \eta \le r\}$, and σ is the ℓ_1-metric on S. In what follows an important role will play a subgraph Γ of $\Gamma_{p,r}$ induced by the nodes (i, j) satisfying $a_j \le i \le b_j$ for two sequences $0 = a_0 \le a_1 \le \ldots \le a_r \le p$ and $0 \le b_0 \le b_1 \le \cdots \le b_r = p$ with $a_j \le b_j$, $j = 0, \ldots, r$. We call Γ a *net*, or an *s–t net*, with the *origin* $s = (0, 0)$ and *end* $t = (p, r)$, and denote by R^Γ (resp. L^Γ) the *rightmost* (resp. *leftmost*) path from s to t in Γ; see Figure 11(b). A node with coordinates (ξ, η) in Γ is denoted by $(\xi, \eta)_\Gamma$.

For a set $V \supseteq T$, a T*-embedding* of V in S is a mapping $\omega : V \to S$ with $\omega(v) = v$ for all $v \in T$. For a u–v path P and nodes $x, y \in P$, $P(x, y)$ denotes the part from x to y in P; P^{-1} denotes the path reverse to P; and $P \cdot Q$ denotes the concatenation of P and a v–w path Q.

THEOREM 4.2. *Let $H = (T, U)$ be a frame, and let (V, m) be a tight extension of H. Then there exists a T-embedding $\omega : V \to S^H$ such that $m(xy) = \sigma^H(\omega(x)\omega(y))$ for all $x, y \in V$. Moreover, such an ω is unique, i.e. (V, m) admits a unique isometric T-embedding in (S^H, σ^H).*

PROOF. It falls into several claims. Claims 1–4 specify the structure of shortest paths and nets in H; these claims will then be used to dispose the elements of V on S and prove the theorem (Claims 5–8). Throughout the proof, d stands for d^H. We essentially use the fact that H does not contain the 'forbidden configurations' $K^-_{3,3}$ and C^+_6 (see Figure 6). This property can be stated as follows:

> if H contains two 4-circuits whose intersection consists of a single edge as drawn in Figure 12, then $\bar{y}$ is the only node in H adjacent to both $\bar{x}$ and $\bar{z}$. (4.1)

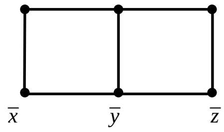

FIGURE 12.

Indeed, the existence of another such node v would cause the appearance of C_6^+ (when v is different from all nodes in H') or $K_{3,3}^-$ (otherwise).

We start from the following basic fact.

CLAIM 1. *For $s, t \in T$, let P and P' be shortest s–t paths in H. Then there is an s–t net Γ in H with $R^\Gamma = P$ and $L^\Gamma = P'$.*

PROOF. Let $P = x_0x_1 \dots x_k$ and $P' = y_0y_1 \dots y_k$; so $s = x_0 = y_0$, $t = x_k = y_k$ and $d(st) = k$. We use induction on k. If $k \leq 2$, the result is obvious; so assume that $k \geq 3$. Also we assume that P and P' have no common inner node, else the result easily follows by induction.

Let P_i stand for $P(x_0, x_i)$ and P_i' stand for $P'(y_0, y_i)$. Since H has no isometric $2k$-circuit, there are $0 < i, j < k$ such that both $i + j$ and $2k - i - j$ are greater than $q = d(x_iy_j)$ (whence $q \leq k - 2$ since H is bipartite). One may assume that such i, j are chosen so that $i + j$ is minimum, that $i + j \leq 2k - i - j$ (otherwise permute s and t), and that $j \leq i$. Let $B = z_0z_1 \dots z_q$ be a shortest path from $z_0 = y_j$ to $z_q = x_i$. Then (i) no inner node of B meets $P_i \cup P_j'$ and (ii) $d(x_iy_{j-1}) = i + j - 1 = q + 1$ (since H is bipartite).

By (ii), $d(x_iy_{j-1}) < k$, and both $y_{j-1} - x_i$ paths $Q = y_{j-1}y_{j-2} \dots y_0x_1 \dots x_i$ and $Q' = y_{j-1}z_0z_1 \dots z_q$ are shortest. By induction there is a $y_{j-1} - x_i$ net Γ' with $R^{\Gamma'} = Q$ and $L^{\Gamma'} = Q'$. Since the path $y_jy_{j-1} \dots y_0x_1 \dots x_{i-1}$ is shortest (by the minimality of $i + j$) and contains $y_{j-1} = (0, 0)_{\Gamma'}$, we have $y_j = (0, 1)_{\Gamma'}$ and $x_{i-1} = (q, 0)_{\Gamma'}$. This implies that $x_i = (q, 1)_{\Gamma'}$. Therefore, Γ' is the grid $\Gamma_{q,1}$ in which $B = (0, 1)(1, 1) \dots (q, 1)$. Also $s = (a, 0)_{\Gamma'}$ for some $0 \leq a \leq q$.

Let D be the s–x_i path of the form $(a, 0)(a, 1)(a + 1, 1) \dots (q, 1)$, and D' be the s–y_j path of the form $(a, 0)(a, 1)(a - 1, 1) \dots (0, 1)$ in Γ'. Since $|D| = |P_i|$ and $|D'| = |P_j'|$, the s–t paths $\bar{P} = D \cdot P(x_i, x_k)$ and $\bar{P}' = D' \cdot P'(y_j, y_k)$ are shortest. Also $s' = (a, 1)_{\Gamma'}$ is a common node in these paths. Let $\tilde{P}(\tilde{P}')$ be the part of $\bar{P}$ (resp. $\bar{P}'$) from s' to t. Then $|\tilde{P}| = |\tilde{P}'| < k$, so by induction there is an s'–t net Γ'' with $R^{\Gamma''} = \tilde{P}$ and $L^{\Gamma''} = \tilde{P}'$. Let y_j and x_i have coordinates (α, β) and (γ, δ) in Γ'', respectively. Note that $j \leq i$ implies $x_i \neq s'$ (otherwise $i = j = 1$ and $q = 0$, whence $x_i = y_j$). The above shortest path B lies in Γ'' and contains s', therefore, $\alpha + \beta + \gamma + \delta = |\alpha - \gamma| + |\beta - \delta|$. This is possible only if $\alpha = 0$ and $\min\{\beta, \delta\} = 0$, taking into account that $\alpha + \beta \leq \gamma + \delta$ (as $j \leq i$).

Case $\beta > 0$ is impossible. For otherwise the subgraph of Γ' induced by the nodes (p, r) for $p = a - 1$, a, $a + 1$ and $r = 0, 1$ together with the path ((0,1),(1,1),(1,0)) in Γ'' violates (4.1). Thus, $\beta = 0$, whence $j = 1$ and $y_1 = s'$. Assume that among all possible net representations for Γ'' (when Γ'' is not 2-edge-connected), the net is chosen so that the coordinate δ is as small as possible. If $\delta = 0$, then the union of Γ' and Γ'' is just the desired s–t net Γ with $R^\Gamma = P$ and $L^\Gamma = P'$. Suppose that $\delta > 0$. Then Γ'' contains nodes $u = (h, 0)$, $v = (h + 1, 0)$, $w = (h + 1, 1)$ and $z = (h, 1)$ such that u, v, w belong to B. But the two 4-circuits in Γ' that contain v together with the path uzw in Γ'' violate (4.1); a contradiction. □

(Note that for $s, t \in T$, the subgraph H' of H that is the union of *all* shortest s–t paths is not necessarily a net; e.g., if $H = H' \simeq K_{2,3}$.) Claim 1 implies:

$$\text{let } \bar{v} \in T, \text{ and let } \bar{x}, \bar{y}, \bar{z} \in T \text{ be distinct nodes such that } \bar{y} \text{ and } \bar{z} \text{ are adjacent to } \bar{x} \text{ and } d(\overline{vy}) = d(\overline{vz}) = d(\overline{vx}) - 1; \text{ then there exists a unique node } \bar{u} \text{ adjacent to both } \bar{y} \text{ and } \bar{z} \text{ for which } d(\overline{vu}) = d(\overline{vx}) - 2. \tag{4.2}$$

Indeed, take a $\bar{v}$–$\bar{x}$ net Γ with R^Γ containing $\bar{y}$ and L^Γ containing $\bar{z}$, and let $\bar{x} = (p, q)_\Gamma$. Then the node $\bar{u} = (p-1, q-1)_\Gamma$ is adjacent to $\bar{y}$ and $\bar{z}$ and satisfies $d(\overline{vu}) = d(\overline{vx}) - 2$. If there are two different nodes u, u' adjacent to both $\bar{y}, \bar{z}$ and closer to $\bar{v}$ than $\bar{y}$, then Γ can be chosen so that $\bar{y}, u \in R^\Gamma$ and $\bar{z}, u' \in L^\Gamma$; without loss of generality, $u = (p, q-2)_\Gamma$. But then $\bar{y} = (p, q-1)_\Gamma$ belongs to two 4-circuits in Γ, and (4.1) is violated because both $\bar{x}$ and u are also adjacent to $\bar{z} \neq \bar{y}$.

REMARK. (4.2) easily implies that S is *contractible* to each $v \in T$, i.e. there is a continuous mapping (homotopy) $h : S \times [0, 1] \to S$ such that $h(\rho, 0) = \rho$ and $h(\rho, 1) = v$ for all $\rho \in S$. The contractibility of the universal tight extension for an arbitrary metric space was shown in [8].

CLAIM 2. *Every 2-connected net in H is isometric.*

PROOF. Suppose this is not so for some 2-connected s–t net Γ. Then there are $x = (p, q)_\Gamma$ and $y = (p', q')_\Gamma$ for which $\Delta := |p - p'| + |q - q'| > d(xy)$. In addition, let x, y be chosen so that Δ is minimum. Since H is bipartite, $\Delta > 2$.

The facts that Γ is 2-connected and $\Delta > 2$ provide the existence of an x–y path of length Δ in Γ, $Q = x_0 x_1 \ldots x_\Delta$ say, such that at least one of $|p_0 - p_2|, |q_0 - q_2|, |p_\Delta - p_{\Delta-2}|, |q_\Delta - q_{\Delta-2}|$ equals two, where $x_i = (p_i, q_i)_\Gamma$. Assume for definiteness that $q_2 = q_0 + 2$; then $q_1 = q_0 + 1$ and $p_0 = p_1 = p_2$. Since Γ is 2-connected, Γ contains nodes $u = (\bar{p}, q_0)$, $v = (\bar{p}, q_1)$ and $w = (\bar{p}, q_2)$, where $\bar{p}$ is either $p_0 - 1$ or $p_0 + 1$.

Let Q' be the part of Q from x_1 to $x_\Delta = y$, and P' be the concatenation of $x_1 x_0$ and a shortest x–y path. Since H is bipartite, the minimal choice of x, y implies that both P' and Q' are shortest x_1–y paths. By Claim 1, there is an x_1–y net Γ' with $R^{\Gamma'} = P'$ and $L^{\Gamma'} = Q'$. Then $x_0 = (1, 0)_{\Gamma'}$, $x_1 = (0, 0)_{\Gamma'}$ and $x_2 = (0, 1)_{\Gamma'}$. But the subgraph of Γ induced by $\{x_0, x_1, x_2, u, v, w\}$ together with the path (1,0)(1,1)(0,1) in Γ' violates (4.1). □

CLAIM 3. *Let $C = v_0 v_1 v_2 v_3 v_0$ be a 4-circuit in H. Let $x, y \in T$ be such that $d(xv_2) = d(xv_0) + 2$ and $d(yv_0) = d(yv_2) + 2$. Then $d(xy) = d(xv_0) + d(yv_2) + 2$, i.e. $v_0 v_1 v_2$ is extended to a shortest x–y path.*

PROOF. Suppose that $d(xy) < d(xv_0) + d(yv_2) + 2\gamma =: \Delta$, and assume that x, y are chosen so that Δ is minimum under this supposition. Take an x–y path P' that is the concatenation of a shortest x–v_0 path, the path $v_0 v_1 v_2$ and a shortest v_2–y path; let $P' = x_0 x_1 \ldots x_k$ and $v_0 = x_i$ (then $v_2 = x_{i+2}$). We have $|P'| = d(xv_0) + d(yv_2) + 2$. Since $d(xv_2) = d(xv_0) + 2$, the subpath $x_0 x_1 \ldots x_{i+2}$ is shortest, whence $k > i + 2$. Similarly, $x_i \ldots x_k$ is shortest, whence $i > 0$. Consider the path $P = x_1 x_2 \ldots x_k$ and the concatenation Q of $x_1 x_0$ and a shortest path from $x_0 = x$ to $x_k = y$. By the minimality of x, y, both paths P and Q are shortest and have no common inner node. Also Q does not meet v_3. By Claim 1, there is an x_1–x_k net Γ with $R^\Gamma = P$ and $L^\Gamma = Q$. We have $x_1 = (0, 0)_\Gamma$ and $x_0 = (0, 1)_\Gamma$. The fact that the path $x_0 x_1 \ldots x_{k-1}$ is shortest (by the minimality of x, y) implies that $x_j = (j-1, 0)_\Gamma$ for $j = 1, \ldots, k-1$. Therefore, $x_k = (j-1, 1)_\Gamma$, i.e., Γ is the grid $\Gamma_{k-2,1}$. Now the two 4-circuits in Γ that share $v_1 = x_{i+1}$ together with the path $v_0 v_3 v_2$ violate (4.1). □

This claim is strengthened as follows:

$$\text{let } \bar{\Gamma} \text{ be a 2-connected } \bar{s}\text{–}\bar{t} \text{ net with } \bar{t} = (p, q)_{\bar{\Gamma}}, \text{ and let } \bar{x}, \bar{y} \in T \text{ be two nodes such that } d(\bar{x}s') = d(\overline{xs}) + 2 \text{ and } d(\bar{y}t') = d(\bar{y}\bar{t}) + 2, \text{ where } s' = (1, 1)_{\bar{\Gamma}} \text{ and } t' = (p-1, q-1)_{\bar{\Gamma}}; \text{ then } d(\overline{xy}) = d(\overline{xs}) + d(\bar{y}\bar{t}) + p + q. \tag{4.3}$$

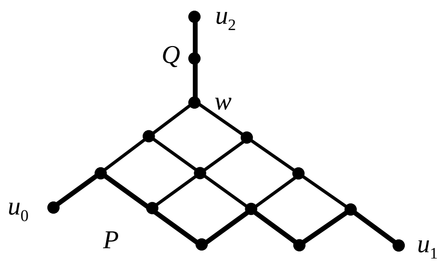

FIGURE 13.

Indeed, since $\bar{\Gamma}$ is isometric (by Claim 2), $d(\bar{s}\bar{t}) = p + q$ and $d(\bar{s}t') = d(s'\bar{t}) = p + q - 2$. Applying Claim 3 to $x = \bar{x}$, $y = \bar{t}$ and the 4-circuit in Γ that contains $\bar{s}$ and s', we have $d(\bar{x}\bar{t}) = d(\overline{xs}) + d(s'\bar{t}) + 2 = d(\overline{xs}) + p + q$, whence $d(\bar{x}\bar{t}) = d(\bar{x}t') + 2$. Now Claim 3 applied to $x = \bar{x}$, $y = \bar{y}$ and the 4-circuit in Γ that contains t' and $\bar{t}$ yields $d(\overline{xy}) = d(\bar{x}t') + d(\bar{y}\bar{t}) + 2$, and (4.3) follows.

CLAIM 4. *Let $u_0, u_1, u_2 \in T$, and let P be a shortest u_0–u_1 path in H. Then there exists a u_0–u_1 net Γ and there exists a path Q from u_2 to some node $w \in \Gamma$ such that: (i) $R^\Gamma = P$; (ii) w is the point $(0, r)$ in Γ, where $u_1 = (p, r)_\Gamma$; (iii) w is the only common node in Q and Γ; and (iv) $\Gamma \cup Q$ is isometric. (See Figure 13.)*

PROOF. Choose $w \in T$ and u_i–w paths B_i, $i = 0, 1, 2$, such that each of $B_0 \cdot B_1^{-1}$, $B_1 \cdot B_2^{-1}$ and $B_2 \cdot B_0^{-1}$ is shortest (they exist by (1.3)). In addition, assume that the w and B_i's are chosen so that the number of common edges in P and $P' = B_0 \cdot B_1^{-1}$ is as large as possible. Then the desired path Q is B_2 and the desired net Γ is a u_0–u_1 net with $R^\Gamma = P$ and $L^\Gamma = P'$, assuming that Γ is chosen so that $\xi + r - \eta$ is as small as possible, where $w = (\xi, \eta)_\Gamma$ and $u_1 = (p, r)_\Gamma$. Property (iii) is obvious, and (ii) easily follows from the maximality of $|P \cap P'|$ and the minimality of $\xi + r - \eta$.

Property (iv) is obvious if P and P' coincide. If $P \neq P'$, consider the subnet Γ' of Γ bounded by the noncommon edges in P and P'. Clearly Γ' contains w and is 2-connected. By Claim 2, Γ' is isometric, which easily implies that Γ is isometric. To obtain (iv), it is enough to show that $d(u_2 v) = d(u_2 w) + d(wv)$ for each $v \in \Gamma$. The latter is true if $v \in P'$ (since $B_2 \cdot B_0^{-1}$ and $B_2 \cdot B_1^{-1}$ are shortest). Let $v \notin P'$ and $v = (\alpha, \beta)_\Gamma$. Then $\alpha > 0$ and $\beta < r$, and now the desired equality follows from Claim 3 regarding u_2, v and the 4-circuit C in Γ that contains w (note that the maximality of $|P \cap P'|$ implies that $d(u_2 z) = d(u_2 w) + 2$, where $z = (1, r - 1)_\Gamma$). $\square$

Next we study tight extensions of H. They admit the following characterization (cf., e.g., [5]):

$$m \in \mathcal{P}_{V,H} \text{ is tight if and only if for any } x, y \in V \text{, there are } s, t \in T \text{ such that } m(sx) + m(xy) + m(yt) = m(st) (= d(st)). \tag{4.4}$$

Indeed, part 'if' is obvious, and if there are $x, y \in V$ with $m(xy) > d(st) - m(sx) - m(ty)$ for all $s, t \in T$, then one can decrease m on some pairs, obtaining a smaller extension of H.

Consider a tight extension (V, m) of H and fix an element $x \in V$. By (4.4), for each $s \in T$, there exists $t \in T$ such that $m(sx) + m(xt) = d(st)$. Claims 3 and 4 enable us to prove the following key fact.

CLAIM 5. *At least one of the following is true:*

(i) $m(v_0x) = 0$ *for some node* v_0 *of* H;
(ii) $m(v_0x) + m(xv_1) = 1$ *for some edge* v_0v_1 *of* H;
(iii) $m(v_0x) + m(xv_2) = 2$ *for some 4-circuit* $C = v_0v_1v_2v_3v_0$ *of* H.

PROOF. Choose $s, t \in T$ such that $k := d(st)$ is minimum provided that $m(sx) + m(xt) = d(st)$, and let $P = z_0z_1 \dots z_k$ be a shortest s–t path in H. Assume that $k \geq 2$ (otherwise (i) or (ii) is true). Consider some $i \in \{1, \dots, k-1\}$, and choose $v \in T$ such that $m(z_ix) + m(xv) = d(z_iv)$. Let Γ, Q, w be as in Claim 4 for $u_0 = s$, $u_1 = t$, $u_2 = v$ and P. We observe that $z_i \notin L^\Gamma$.

Indeed, suppose that $z_i \in L^\Gamma$. Then we may assume without loss of generality that $L^\Gamma = P$ and $w = z_j$ for some $i \leq j \leq k$. Let D be the concatenation of $z_iz_{i+1} \dots z_j$ and Q^{-1}. Then D is shortest, and we have $m(z_ix)+m(xv) = |D| = |Q|+j-i$. Also $m(sx)+m(xt) = k$ and $m(sx)+m(xv) \geq d(sv) = |Q|+j$. Putting together these relations, we get $m(z_ix)+m(xt) \leq k-i = d(z_it)$. This implies $m(z_ix) + m(xt) = d(z_it) < d(st)$, contrary to the choice of s, t.

Applying this observation to $i = 1$ and letting $t = (p, r)_\Gamma$ (whence $w = (0, r)_\Gamma$), we have $z_1 = (1, 0)_\Gamma$ and $r \geq 1$. We may assume that the above P is chosen so that $z_j = (1, j-1)_\Gamma$ for $j = 1, \dots, r+1$ (and $z_j = (j-r, r)_\Gamma$ for $j = r+1, \dots, k$). This means that Γ is the union of a $1 \times r$ grid Γ^1 (with the origin s and end z_{r+1}) and the path $z_{r+1} \dots z_k$. If $k = 2$, we obtain (iii); so assume that $k \geq 3$.

Fix the node $z' = (0, 1)_\Gamma$, take the path $P' = z_0z'z_2 \dots z_k$ and choose $v' \in T$ such that $m(z'x) + m(xv') = d(z'v')$. Let Γ', Q', w' be as in Claim 4 for $u_0 = s$, $u_1 = t$, $u_2 = v'$ and P'. Let $t = (p', r')_{\Gamma'}$; then $w' = (0, r')_{\Gamma'}$. By the above argument, $z' = (1, 0)_{\Gamma'}$ and $r' \geq 1$. We assert that

$$z_i = (i-1, 1)_{\Gamma'} \qquad \text{for } i = 2, \dots, r+1. \tag{4.5}$$

Indeed, if $z_2 \neq (1, 1)_{\Gamma'}$, then $z_2 = (2, 0)_{\Gamma'}$, and the two 4-circuits in Γ' that contain z' together with the path $z_0z_1z_2$ violate (4.1). Similarly, if there is some $2 < i \leq r$ such that $z_i = (i-1, 1)_{\Gamma'}$ but $z_{i+1} = (i-1, 2)_{\Gamma'}$, then (4.1) is violated by the two 4-circuits in Γ that contain z_i and the path $z_{i-1}uz_{i+1}$ with $u = (i-2, 2)$ in Γ' (note that $z_{i-1} = (i-2, 1)_{\Gamma'}$). Hence, (4.5) is true.

In view of (4.5), we may assume that $z_j = (r, j-r)_{\Gamma'}$ for $j = r+1, \dots, r+r'$ (for we can vary the part of P from z_{r+2} to z_{k-1}). In other words, the part Γ^2 of Γ' between s and $z_{r+r'}$ is an $r \times r'$ grid from which the nodes $(2, 0), \dots, (r, 0)$ are removed.

We now combine the Γ^2 with the (transposition of the) above Γ^1 into one $r \times r'$ grid as follows. Observe that $z_0, z', z_2, \dots, z_{r+1}$ are common nodes in Γ^1 and Γ^2, while $z_1 (= (1, 0)_\Gamma)$ may differ from the node $z'' = (0, 1)$ in Γ^2. No other nodes in Γ^1 and Γ^2 can coincide. For if $(0, \eta)_\Gamma = (\alpha, \beta)_{\Gamma'}$ for some $2 \leq \eta \leq r$, then the nodes $y = (\alpha, \beta)$ and $(\eta, 1)$ of Γ' are connected by an edge (which is the edge between $(0, \eta)$ and $(1, \eta)$ in Γ). Since Γ' is isometric, this is possible only if $\alpha = \eta$ and $\beta = 2$. Then the distance between s and y in Γ' is $\eta + 2$, while the distance between these nodes in Γ is η; a contradiction. By a similar reason, z_1 can coincide only with z''. Therefore, the union of $\Gamma^1 - \{z_1\}$ and Γ^2 is indeed an $r \times r'$ grid Γ''. It is convenient to assume that the coordinates in Γ'' are given so that $w = (0, 0)_{\Gamma''}$, $w' = (r, r')_{\Gamma''}$ and $s = (r, 0)_{\Gamma''}$; then $z' = (r-1, 0)_{\Gamma''}$ and $z'' = (r, 1)_{\Gamma''}$.

Let $u = (1, 1)_{\Gamma''}$, $u' = (r-1, r'-1)_{\Gamma''}$, $q = |Q|$ and $q' = |Q'|$. Since $\Gamma' \cup Q'$ is isometric, $w' = (0, r')_{\Gamma'}$ and $u' = (1, r'-1)_{\Gamma'}$, we have

$$d(v'u') = q' + 2. \tag{4.6}$$

Similarly, if $r > 1$, then the facts that $\Gamma \cup Q$ is isometric and $u = (1, r-1)_\Gamma$ imply

$$d(vu) = q + 2. \tag{4.7}$$

Note that (4.7) remains true for $r = 1$. Indeed, $r = 1$ implies that $w = z'$, $u = z''$, $d(vz_1) = q + 2$ and $d(vs) = q + 1$; therefore, if $d(vu) \neq q + 2$, then $u \neq z_1$ and $d(vu) = q$.

This contradicts (4.2) because $d(vz_1) = q + 2$, z_1 is adjacent to the nodes s and z_2 with $d(vs) = d(vz_2) = q + 1$, and each of s, z_2 is adjacent to the distinct nodes w and u with $d(vw) = d(vu) = q$.

Finally, in view of (4.6) and (4.7), we can apply (4.3) to Γ'', v, v' and conclude that $d(vv') = q + q' + r + r'$. Now the relations

$$m(vx) + m(xz_1) = d(vz_1) = q + r + 1,$$
$$m(v'x) + m(xz') = d(v'z') = q' + r' + 1,$$
$$m(vx) + m(xv') \geq d(vv') = q + q' + r + r'$$

yield $m(z_1x) + m(xz') \leq 2$. This gives (iii) (with $C = z_1sz'z_2z_1$). □

We call x a 0-*point* (1-*point*; 2-*point*) if it satisfies (i) (resp. (ii) but not (i); (iii) but not (ii)) in Claim 5. If x is a 0-point or 1-point, then its image $\omega(x)$ in S is defined in a natural way. More precisely, if $m(vx) = 0$ for some $v \in T$, define $\omega(x) = v$; this $\omega(x)$ is a unique point x' on S such that $\sigma(vx') = m(vx)$. Similarly, if $m(ux) + m(xv) = 1$ for some $e = uv \in U$, define $\omega(x)$ to be the point x' on e with $\sigma(ux') = m(ux)$; one can see that this x' is a unique point on S satisfying $\sigma(ux') = m(vx)$ and $\sigma(vx') = m(ux)$. Note also that if there is another edge $u'v'$ with $m(u'x) + m(xv') = 1$, then assuming without loss of generality that $d(uu') \geq 2$, we have $m(vx) + m(v'x) = 2 - m(ux) - m(u'x) \leq 0$, whence $m(vx) = 0$, i.e. x is a 0-point.

The next claim points out a cell where $\omega(x)$ should be disposed when x is a 2-point. For $p, q \in T$, let $T(p, q)$ denote the set of nodes of H adjacent to both p and q.

CLAIM 6. *Let x be a 2-point, and let $m(px) + m(xq) = 2$ for opposite nodes p and q in some 4-circuit. Let $z \in T(p, q)$. Then there is $v \in T(p, q) - \{z\}$ such that $m(vx) + m(xz) = 2$.*

PROOF. Choose $u' \in T$ such that $m(u'x) + m(xz) = d(u'z) =: k$. Since H is bipartite, $d(u'p)$ is $k - 1$ or $k + 1$. If $d(u'p) = k + 1$, then $m(u'x) + m(xp) \geq k + 1$ together with $m(u'x) + m(xz) = k$ and $m(qx) + m(xp) = 2$ implies $m(qx) + m(xz) \leq 1$, which is impossible as x is a 2-point. Hence, $d(u'p) = k - 1$. Similarly, $d(qu') = k - 1$. By (4.2), p and q are adjacent to a node v with $d(u'v) = k - 2$.

Next, choose $v' \in T$ such that $m(v'x) + m(xv) = d(v'v) =: \ell$. Arguing as above, we have $d(v'u) = \ell - 2$ for some $u \in T(p, q)$. Also $d(u'u) = k$ (otherwise we would have $d(u'u) = d(u'v) = k - 2$, and (4.2) is violated because $d(u'z) = k$ and $d(u'p) = d(u'q) = k - 1$). Thus, we can apply Claim 3 to the 4-circuit on u, p, v, q and the nodes u', v', yielding $d(u'v') = d(u'v) + d(uv') + 2 = k + \ell - 2$. Hence, $m(u'x) + m(xv') \geq k + \ell - 2$. Comparing this inequality with $m(u'x) + m(xz) = k$ and $m(vx) + m(xv') = \ell$, we get $m(vx) + m(xz) \leq 2$, and the result follows. □

By Claim 6, if x is a 2-point, then there is a 4-circuit $C = v_0v_1v_2v_3v_0$ such that $m(v_ix) + m(xv_{i+2}) = 2$ for $i = 0, 1$. We say that C *surrounds* x. Define $\omega(x)$ to be the point (ξ, η) in D^C with $\xi + \eta = m(v_0x)$ and $\xi + 1 - \eta = m(v_1x)$ (letting $v_0 = (0, 0)$ and $v_1 = (0, 1)$). We have to show that such a position of $\omega(x)$ on S is well-defined.

CLAIM 7. *(i) If $C' = u_0u_1u_2u_3u_0$ is another 4-circuit surrounding x, and $\omega'(x)$ is defined as above with respect to C', then C and C' belong to the same 2-clique, and $\omega'(x) = \omega(x)$. (ii) $x' = \omega(x)$ is the only point on S with $\sigma(v_ix') = m(v_ix)$ for $i = 0, 1, 2, 3$.*

PROOF. Denote $m(u_ix)$ by α_i and $m(v_ix)$ by β_i. Let $\{T', T''\}$ be the partition of T into stable sets in H.

First, suppose that C and C' are disjoint. One may assume that $u_0, v_0 \in T'$. Then $u_2, v_2 \in T'$ and $u_1, u_3, v_1, v_3 \in T''$. This implies $\alpha_i + \beta_i \geq d(u_iv_i) \geq 2$ for $i = 0, 1, 2, 3$; moreover, equality must hold throughout since the sum Σ of all α_i's and β_j's amounts to eight. Similarly,

$\alpha_i + \beta_{i+2} = 2$ (taking indices modulo 4). This gives $\alpha_i = \beta_i = 1$, in view of $\beta_i + \beta_{i+2} = 2$. Therefore, $d(u_i v_j) \le \alpha_i + \beta_j = 2$ for all i, j, whence the subgraph of H induced by $\{u_0, v_0, \dots, u_3, v_3\}$ is isomorphic to $K_{4,4}$; a contradiction.

Second, suppose that C and C' have an only common node, $u_0 = v_0 \in T'$ say. We may assume that $d(u_2v_1) = d(u_1v_2) = 3$ (taking into account that if $d(u_2v_1) = d(u_2v_3) = 1$, then H contains $K^-_{3,3}$, and similarly for v_2). Therefore, $\alpha_2 + \beta_1 \ge 3$ and $\alpha_1 + \beta_2 \ge 3$. Also $\alpha_0 + \beta_0 \ge 0$ and $\alpha_3 + \beta_3 \ge 2$. Since $\Sigma = 8$, equality holds throughout. But $\beta_0 = 0$ means that x is a 0-point; a contradiction.

Third, suppose that C and C' have exactly two common nodes and these nodes are adjacent, $u_0 = v_0$ and $u_1 = v_1$ say. Then $d(u_2v_3) = d(u_3v_2) = 3$ (otherwise H would contain $K^-_{3,3}$). Therefore, $\alpha_2 + \beta_3 \ge 3$ and $\alpha_3 + \beta_2 \ge 3$; also $\alpha_0 + \alpha_1 \ge 1$ and $\beta_0 + \beta_1 \ge 1$. This implies $\beta_0 + \beta_1 = 1$, i.e. x is a 1-point; a contradiction.

Thus, C and C' have a pair of opposite nodes in common, $u_0 = v_0$ and $u_2 = v_2$ say. If the other nodes are different, then $\alpha_1 + \alpha_3 = \beta_1 + \beta_3 = 2$ and $\alpha_i + \beta_j \ge 2$ for $i, j = 1, 3$ imply $\alpha_q = \beta_q = 1$ for $q = 1, 3$. And if $u_1 = v_1$ say, then $\alpha_3 = \beta_3 = 2 - \alpha_1 \ge 1$. In both cases, $\omega(x)$ and $\omega'(x)$ are the same point in the folder involving D^C and $D^{C'}$.

To see (ii), use the argument as in the proof of Statement 4.1 and observe that if $x'' \in S$ is not in the folder including D^C, then at least one of $\sigma(v_0x'') + \sigma(x''v_2)$ and $\sigma(v_1x'') + \sigma(x''v_3)$ is greater than two. □

For a 2-point x, let $H(x)$ denote the subgraph of H that is the union of all 4-circuits surrounding x. Claims 6 and 7 show that $H(x)$ is a 2-clique (so $\omega(x)$ is an interior point of the folder $F^{H(x)}$). We know that σ coincides with $\sigma^{H'}$ within each folder. In view of Claim 6,

if x is a 2-point and $\{T_1, T_2\}$ is the partition of the nodes of $H(x)$ into stable sets, then: (i) $\sigma(v\omega(x)) = m(vx)$ for each node v of $H(x)$; and (ii) for $i = 1, 2$, there is $0 < \varepsilon \le 1/2$ such that $m(ux) = \varepsilon$ for some $u \in T_i$ and $m(vx) = 2 - \varepsilon$ for all other $v \in T_i$; in particular, if $v_0v_1v_2v_3v_0$ is a 4-circuit surrounding x, and $\tilde{v} \in T$ is adjacent to both v_0, v_2, then at least one of $v_0\tilde{v}v_2v_3v_0$ and $v_0v_1v_2\tilde{v}v_0$ is a 4-circuit surrounding x. (4.8)

Our final claim shows that ω is indeed an isometric embedding for (V, m).

CLAIM 8. *Let $x, y \in V$, $x' = \omega(x)$ and $y' = \omega(y)$. Then $\sigma(x'y') = m(xy)$.*

PROOF. First we prove that

$$\text{for any } s \in T, m(sx) = \sigma(sx') \text{ (and similarly, } m(sy) = \sigma(sy')). \tag{4.9}$$

This is so if x is a 0-point, by Statement 4.1. Assume that x is a 2-point (if x is a 1-point, the proof of (4.9) is technically simpler, and we leave it to the reader). Consider $s \in T$ and choose $t \in T$ such that $m(sx) + m(xt) = m(st)(= d(st))$ (existing as m is tight). To prove (4.9), we show that

there exists a 4-circuit $C = upvqu$ in $H(x)$ surrounding x and such that the path $suxvt$ on V is shortest for m, i.e. $m(su) + m(ux) + m(xv) + m(vt) = m(st)$. (4.10)

To see this, take an arbitrary 4-circuit $B = u_0u_1u_2u_3u_0$ surrounding x and assume for definiteness that $\max\{m(su_i) : i = 0, 1, 2, 3\} = m(su_2) =: k$. Then $m(su_1) = m(su_3) = k - 1$ and $m(su_0) \in \{k - 2, k\}$. Without loss of generality one may assume that $m(su_0) = k - 2$. Indeed, if $m(su_0) = k$, then there exists $z \in T$ adjacent to both u_1, u_3 for which $m(sz) = k - 2$ (by (4.2) for s, u_2, u_1, u_3), and at least one of the 4-circuits $zu_1u_2u_3z$ and $zu_1u_0u_3z$ surrounds x (by (4.8)); so we can replace B by this circuit. The equalities $m(su_0) = k - 2, m(su_2) = k$ and $m(u_0x) + m(xu_2) = 2$ imply that the path su_0xu_2 is shortest for m. Similarly, there exists a 4-circuit $B' = v_0v_1v_2v_3v_0$ surrounding x and such that the path tv_2xv_0 is m-shortest. Now $m(sx)+m(xt) = m(st)$ implies that the path su_0xv_2t is m-shortest, whence $m(u_0x)+m(xv_2) = m(u_0v_2)$.

Note that $m(u_0v_2) = 2$ (because B and B' belong to the same 2-clique $H(x)$, by Claim 7, and $m(u_0v_2) \in \{0, 1\}$ would imply that x is a 0- or 1-point). Hence, u_0, u_2, v_0, v_2 are in the same stable set in $H(x)$. If $\{u_0, u_2\} \neq \{v_0, v_2\}$, then $\{u_1, u_3\} = \{v_1, v_3\}$, and the circuit $u_0u_1v_2u_3u_0$ satisfies (4.10) (with $u = u_0$ and $v = v_2$). And if $\{u_1, u_3\} \neq \{v_1, v_3\}$, then $u_0 = v_0$ and $u_2 = v_2$, and (4.10) holds with $C = B$.

Now (4.10) gives (4.9) as follows. By Statement 4.1, $m(su) = \sigma(su)$ and $m(tv) = \sigma(tv)$. Also $m(ux) = \sigma(ux')$ and $m(vx) = \sigma(vx')$, by (4.8). Therefore, the equality in (4.10) implies $\sigma(sx') \leq \sigma(su) + \sigma(ux') = m(sx)$ and $\sigma(tx') \leq \sigma(tv) + \sigma(vx') = m(tx)$; moreover, these inequalities turn into equalities because $\sigma(sx') + \sigma(x't) \geq m(st)$.

(When x is a 1-point whose image x' lies on an edge uv, one shows that, up to permutation of u and v, the path $suxvt$ is m-shortest; this implies (4.9) is a similar way.)

The claim immediately follows from (4.9) if some of x, y is a 0-point. Assume that both x, y are 2-points (the proof for the other cases relies on similar arguments). Take $s, t \in T$ such that the path $P = sxyt$ on V is m-shortest (cf. (4.4)). Since $m(sx) = \sigma(sx')$, $m(yt) = \sigma(y't)$ and $\sigma(sx') + \sigma(x'y') + \sigma(y't) \geq m(st)$, we have $m(xy) \leq \sigma(x'y')$. To show the reverse inequality, we use (4.10) (for x and y) and consider a 4-circuit $C = upvqu$ surrounding x and a 4-circuit $C' = u'p'v'q'u'$ surrounding y such that

$$\text{the paths } Q = suxvt \text{ and } Q' = sv'yu't \text{ on } V \text{ are } m\text{-shortest.} \tag{4.11}$$

Since the paths P, Q, Q' are shortest, the path $uxyu'$ is shortest too. Let $k = m(uu')$. One may assume that $m(uv') = m(vu') = k - 2$ (because we can apply (4.10) to u, u' in place of s, t). If $k = 2$, then $u = v'$ and $v = u'$; so we may assume that $C = C'$, whence $m(xy) = \sigma^C(x'y') = \sigma(x'y')$. Let $k \geq 3$. Since $m(uv') = k - 2$ and $m(uv) = 2$, we have $m(vv') \in \{k, k-2, k-4\}$. Consider these cases.

Case 1. $m(vv') = k - 4$. Then the path $uxvv'yu'$ is m-shortest. This and the relations $\sigma(x'v) = m(xv)$, $\sigma(vv') = m(vv')$, $\sigma(v'y') = m(v'y)$ and $\sigma(x'y') \leq \sigma(x'v)+\sigma(vv')+\sigma(v'y')$ imply $\sigma(x'y') \leq m(xy)$.

Case 2. $m(vv') = k$. By Claim 1, there is a u–u' net Γ with R^Γ containing v', p' and L^Γ containing p, v. Observe that $m(uu') = m(vv') = k$ is possible only if $v = (0, 2)_\Gamma$ and $v' = (k-2, 0)_\Gamma$, i.e. Γ is a $(k-2) \times 2$ grid. Since $k \geq 3$, the node $p = (0, 1)_\Gamma$ belongs to two 4-circuits in Γ, which together with the path uqv violate (4.1).

Case 3. $d(vv') = k-2$. Recall that $m(uv') = k-2$. Let $a = m(pv')$ and $b = m(qv')$; clearly $a, b \in \{k-3, k-1\}$. Without loss of generality one may assume that $b = k-3$. Indeed, if both a, b equal $k-1$, take the node z adjacent to both u, v and such that $m(zv') = k-3$ (considering (4.2) for v', p, u, v); since at least one of the circuits $upvzu$ and $uqvzu$ surrounds x (by (4.8)), we can replace C by this circuit. Similarly, we may assume that $m(vq') = k - 3$. Then the paths $uqv'p'u'$ and $upvq'u'$ are shortest. Take a u–u' net Γ with R^Γ containing q, v', p' and L^Γ containing p, v, q'. Then $p = (0, 1)_\Gamma$, $q = (1, 0)_\Gamma$, $p' = (\alpha, \beta-1)_\Gamma$ and $q' = (\alpha-1, \beta)_\Gamma$, where $u' = (\alpha, \beta)_\Gamma$. Since $m(vq) = 1$, $v = (1, 1)_\Gamma$; similarly, $v' = (\alpha - 1, \beta - 1)_\Gamma$. Now $m(uu') = k$ and $m(vv') = k - 2$ imply $\beta = 1$, whence $\alpha = k - 1$. Therefore, Γ is a $(k - 1) \times 1$ grid. Finally, let x' and y' have coordinates (ξ, η) and (ξ', η'), respectively, in the region S^Γ of S, i.e., $m(ux) = \xi + \eta$, $m(px) = 1 - \eta + \xi$, $m(u'y) = k - \xi' - \eta'$ and $m(p'y) = k - 1 - \xi' + \eta'$ (taking into account that x' and y' lie in the first cell D^C and the last cell $D^{C'}$ of S^Γ, respectively). Then $\sigma(x'y') \leq \sigma^\Gamma(x'y') = \xi' - \xi + |\eta' - \eta|$ (since $\xi' > \xi$,

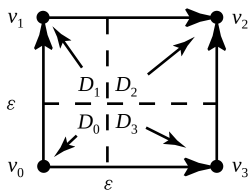

FIGURE 14.

by $k \geq 3$). On the other hand, the m-length of the path $uxyu'$ is equal to k and the m-length of $pxyp'$ is at least k (since $d^\Gamma(pp') = k$ and Γ is isometric, by Claim 2), whence

$$m(xy) = k - m(ux) - m(yu') = k - (\xi + \eta) - (k - \xi' - \eta') = \xi' - \xi + \eta' - \eta; \text{ and}$$
$$m(xy) \geq k - m(px) - m(yp') = k - (1 - \eta + \xi) - (k - 1 - \xi' + \eta') = \xi' - \xi + \eta - \eta'.$$

This gives $m(xy) \geq \xi' - \xi + |\eta' - \eta|$, yielding $\sigma(x'y') \leq m(xy)$. □

This completes the proof of Theorem 4.2. □□

To finish the proof of part (ii) ⇒ (i) in Theorem 1.1, it remains to consider a finite set $V \supseteq T$ on S and show that the metric $\sigma_{|V}$ is H-decomposed by some 0-extension of H. In view of (2.3), it suffices to construct a mapping $V \to T$ which is identical on T and brings any σ-shortest T-path on V to a shortest path on T (this is possible even if V is countable).

For $x \in S$, let $D(x)$ be the cell of S that contains x (or one of such cells if many), and let $(\xi(x), \eta(x))$ be the coordinates of x in $D(x)$. Since V is finite, there exists a number $0 < \varepsilon < 1$ such that, for all $x \in V$, each of $\xi(x)$ and $\eta(x)$ is different from ε and from $1 - \varepsilon$.

Choose a feasible orientation of H (according to Definition 2 in the Introduction). One may assume that for all 4-circuits C of H, the coordinates in D^C are given so that the edge between (0,0) and (0,1) is oriented from (0,0) to (0,1), and the edge between (0,0) and (1,0) is oriented from (0,0) and (1,0) (see Figure 2 where $v_0 = (0, 0)$ and $v_1 = (0, 1)$). Define the following four regions in D^C (see Figure 14):

$$\begin{aligned} D_0 &= \{\{\xi, \eta\} : \xi, \eta < \varepsilon\}, \\ D_1 &= \{\{\xi, \eta\} : \xi < \varepsilon < \eta\}, \\ D_2 &= \{\{\xi, \eta\} : \xi, \eta > \varepsilon\}, \\ D_3 &= \{\{\xi, \eta\} : \eta < \varepsilon < \xi\}. \end{aligned}$$

By the choice of ε, each element of V occurring in D^C is contained in exactly one D_i, and each D_i contains one terminal, denoted by h_i (in Figure 14, $h_i = v_i$). For $i = 0, 1, 2, 3$ and $x \in D_i$, define $\gamma^C(x) = h_i$. Putting the γ^C's together for all 4-circuits C, we get a mapping γ of the union of all regions as above to T which is identical on T. One can see that γ is well-defined (e.g. considering a feasible orientation of the edges of a 2-clique H', observe that the γ^C's are compatible on common parts of cells C in the folder $F^{H'}$). Also γ is *monotonic* in the sense that if $x, y \in D^C$ are such that either $\xi(x) \leq \xi(y) \leq \xi(\gamma(x))$ or $\xi(x) \geq \xi(y) \geq \xi(\gamma(x))$, then $\xi(\gamma(y)) = \xi(\gamma(x))$, and similarly for the second coordinate η. This γ gives the desired mapping of V to T.

For a sequence $\mathcal{P} = x_0 x_1 \ldots x_k$ of points of S (a path on S), its length $\sigma(\mathcal{P})$ is $\sum(\sigma(x_{i-1}x_i) : i = 1, \ldots, k)$; and $\mathcal{P}$ is shortest if $\sigma(\mathcal{P}) = \sigma(x_0 x_k)$.

LEMMA 4.3. *Let $\mathcal{P} = x_0 x_1 \ldots x_k$ be a shortest path on S with $x_0, x_k \in T$, and for $i = 0, \ldots, k$, let $\xi(x_i), \eta(x_i) \neq \varepsilon, 1 - \varepsilon$. Then the path $\gamma(\mathcal{P}) = \gamma(x_0)\gamma(x_1) \ldots \gamma(x_k)$ is also shortest.*

PROOF. Suppose this is not so and choose a counterexample $\mathcal{P} = x_0x_1 \dots x_k$ (without repeated elements) with $\sigma(\mathcal{P})$ minimum. Then none of $x_1, \dots, x_{k-1}$ is in T. We show that there exists a counterexample $\mathcal{P}'$ with $\sigma(\mathcal{P}') < \sigma(\mathcal{P})$, which leads to a contradiction because the lemma is obvious when $\sigma(\mathcal{P})$ is 0 or 1. One may assume that for $i = 1, \dots, k$, x_{i-1} and x_i belong to the same cell (otherwise replace the part $x_{i-1}x_i$ by a shortest path $\mathcal{Q} = y_0y_1 \dots y_r$ from $y_0 = x_{i-1}$ to $y_r = x_i$ in which each pair y_{j-1}, y_j is in the same cell; such a $\mathcal{Q}$ can be chosen so that $\xi(y_j), \eta(y_j) \neq \varepsilon, 1-\varepsilon$ for $j = 0, \dots, r$).

Consider a maximal part $\mathcal{L} = x_0x_1 \dots x_q$ of $\mathcal{P}$ such that all $x_0, \dots, x_q$ belong to the same cell, D^C say. Assume that $C = v_0v_1v_2v_3v_0$, that $x_0 = v_0$, that x_q lies on the edge v_1v_2, and that v_0 and v_1 have the coordinates (0,0) and (0,1) in D^C, respectively (cases $x_0 = v_i$ for $i = 1, 2, 3$ are examined similarly). Let (ξ_i, η_i) be the coordinates of x_i in D^C, $i = 0, \dots, q$ (so $\xi_0 = \eta_0 = 0$ and $\eta_q = 1$). Since $\mathcal{L}$ is shortest, $\xi_0 \le \xi_1 \le \dots \le \xi_q$ and $\eta_0 \le \eta_1 \le \dots \le \eta_q$. Obviously, $\gamma(x_0) = v_0$, $\gamma(x_q) \in \{v_1, v_2\}$ and $\gamma(x_i) \in \{v_0, v_1, v_2, v_3\}$ for $i = 1, \dots, q-1$. Moreover, the monotone property provides $\xi'_0 \le \xi'_1 \le \dots \le \xi'_q$ and $\eta'_0 \le \eta'_1 \le \dots \le \eta'_q$, where $\xi'_i = \xi(\gamma(x_i))$ and $\eta'_i = \eta(\gamma(x_i))$. Hence, the path $\gamma(\mathcal{L})$ is shortest, and deletion of $x_1, \dots, x_{q-1}$ from $\mathcal{P}$ makes again a minimal counterexample. So we may assume that $q = 1$.

Let $\mathcal{P}'$ be the path obtained from $\mathcal{P}$ by inserting v_1 between x_0 and x_1. Obviously, the path $x_0v_1x_1$ is shortest, whence $\mathcal{P}'$ is shortest as well. Hence, $\mathcal{P}'$ is again a minimal counterexample. But $\mathcal{P}'$ contains the terminal v_1 as an intermediate element; a contradiction. □

Thus, $\sigma_{|V}$ is decomposed by the 0-extension to V determined by γ. This completes the proof of Theorem 1.1. □□□

5. RELATIONSHIP TO THE MULTIFLOW LOCKING PROBLEM

There is a large subclass of frames $H = (T, U)$ for which the minimizability can be proved significantly simpler by use of the so-called multiflow locking theorem.

Suppose one is given a finite set V, a function $c : E_V \to \mathbb{Z}_+$ (of edge *capacities*), a set $T \subseteq V$ of terminals, and a collection $\mathcal{A} = \{A_1, \dots, A_p\}$ of subsets of T. By a *multiflow* (multicommodity flow) f one means a set of T-paths $P_1, \dots, P_k$ along with real weights $\lambda_1, \dots, \lambda_k \ge 0$ such that for each $e \in E_V$, the sum of weights λ_i of the paths P_i going through e does not exceed $c(e)$. For $A \subseteq T$, let $\phi^f(A)$ denote $\sum(\lambda_i : i = 1, \dots, k$, P_i has exactly one end in $A)$, and $c\langle A\rangle$ denote the minimum capacity $c(\delta(X)) = \sum(c(e) : e \in \delta(X))$ of a *cut* $\delta(X)$ in (V, E_V) separating A and $T - A$. (For $G = (V, E)$ and $X \subseteq V$, $\delta(X)$ is the set of edges $e \in E$ with one end in X and the other in $V - X$; a cut $\delta(X)$ *separates* sets $A, B \subset V$ if either $A \subseteq X \subseteq V - B$ or $B \subseteq X \subseteq V - A$.) Clearly $\phi^f(A) \le c\langle A\rangle$, and f is said to *lock* A if this turns into equality. The *multiflow locking problem* is to find a multiflow that locks all $A_j \in \mathcal{A}$. If this problem has a solution for all V and c, $\mathcal{A}$ is called *lockable*. Sets $A, B \subseteq T$ are called *crossing* if none of $A \cap B$, $T - (A \cup B)$, $A - B$, $B - A$ is empty. The multiflow locking theorem is as follows.

THEOREM 5.1 ([13]). *(see also [9,16]). $\mathcal{A}$ is lockable if and only if $\mathcal{A}$ is* 3-cross-free, *i.e. no three members of $\mathcal{A}$ are pairwise crossing.*

Among applications of the locking problem, one mentioned in [13] is important to us. It concerns a relationship between the multiflow locking problem and problem (1.2). Given a metric μ on T, the *μ-value* of multiflow $f = (P_1, \dots, P_k; \lambda_1, \dots, \lambda_k)$ is $\sum(\lambda_i\mu(s_it_i) :$ $i = 1, \dots, k, s_i$ and t_i are the ends of $P_i)$. Let $\nu^*(\mu, c)$ denote the maximum μ-value of a multiflow. One can see that (1.2) and the problem:

$$\text{Find a multiflow for } V, T, c \text{ whose } \mu\text{-value is maximum,} \tag{5.1}$$

are dual each other, i.e. $\nu^*(\mu, c) = \tau^*(\mu, c)$. We are interested in a special case in which μ is non-negative linear combination $\alpha_1\mu_1 + \dots + \alpha_p\mu_p$ of *cut metrics* $\mu_j = \rho^{A_j}$ on T, where $A_j \subset T$, and ρ^{A_j} takes value one on each pair st with $|\{s,t\} \cap A_j| = 1$, and zero otherwise. Suppose that the locking problem for c and this $\mathcal{A} = \{A_1, \dots, A_p\}$ has a solution f. It is easily shown that the μ-value of f amounts to the volume $c \cdot m$ for the extension $m = \alpha_1 \rho^{X_1} + \cdots + \alpha_p \rho^{X_p}$ of μ to V, where each ρ^{X_j} is the cut metric on V corresponding to a minimum cut $\delta(X_j)$ separating A_j and $T - A_j$ (i.e. $c(\delta(X_j)) = c\langle A_j \rangle$). Therefore, $c \cdot m = \tau^*(\mu, c)$. As a consequence (in view of Theorem 5.1), if $\mathcal{A}$ is 3-cross-free and all α_j's are ones, then the metric $m = \rho^{X_1} + \cdots + \rho^{X_p}$ with X_j's constructed this way always gives an optimal solution to (1.2).

For certain frames H the distance function $\mu = \mathrm{d}^H$ is represented as the sum of cut metrics whose inducing sets form a 3-cross-free family and, moreover, the above extension m is a 0-extension. This relies on the following result (details of the proof are left to the reader).

THEOREM 5.2. *Let $H = (T, U)$ be a frame without subgraphs $K_{2,3}$. Let $U_1, \dots, U_p$ be the orbits of H (defined as in Section 2 for $G = H$). Then:*

- *(i) each U_j is a cut $\delta(A_j)$ in H;*
- *(ii) the set $\mathcal{A} = \{A_1, \dots, A_p\}$ is 3-cross-free;*
- *(iii) for any $s, t \in T$, $\mathrm{d}^H(st)$ equals the number of cuts $\delta(A_j)$ separating s and t; in other words, $\mathrm{d}^H = \rho^{A_1} + \cdots + \rho^{A_p}$;*
- *(iv) the family $\bar{\mathcal{A}} = \{A_1, \dots, A_p, T - A_1, \dots, T - A_p\}$ possesses the property that for any subfamily of $\bar{\mathcal{A}}$, its members have a common element provided that they are pairwise intersecting.*

SKETCH OF PROOF. Consider a dual path $D = (e_0, F_1, e_1, \dots, F_k, e_k)$ with $e_i = x_i y_i$ and $F_i = x_{i-1} y_{i-1} y_i x_i x_{i-1}$ (defined in Section 3). Then for any $0 \le i \le j \le k$,

$$\mathrm{d}^H(x_i x_j) = \mathrm{d}^H(y_i y_j) = \mathrm{d}^H(x_i y_j) - 1 = \mathrm{d}^H(y_i x_j) - 1 = j - i. \tag{5.2}$$

Indeed, if $\mathrm{d}^H(x_i x_j) < j - i$ for some i, j and $j - i$ is minimum, then $\mathrm{d}^H(x_i x_j) = j - i - 2$ and $j > i + 2$ (otherwise H contains $K_{2,3}$). Considering an $x_{i+1} - x_j$ net Γ with $R^\Gamma = x_{i+1} \dots x_j$ and L^Γ to be a shortest x_{i+1}–x_j path containing x_i, and taking the node (1.1) in Γ, we get a contradiction with (4.1). The remaining equalities in (5.2) are obtained in a similar way. A consequence of (5.2) is that all $x_0, \dots, x_k, y_0, \dots, y_k$ are different. Next, since H has no isometric circuit of size six or more, any two edges in an orbit U_j are connected by a dual path. The set of ends of edges in U_j is partitioned into two subsets R_j and L_j so that the nodes of each subset lie on the same side of dual paths (i.e. all being of the form x_i or of the form y_i).

Suppose that some U_j is not a cut. Then there is a path P in $(T, U - U_j)$ from $x \in R_j$ to $y \in L_j$. One may assume that x and y are the ends of the same edge e and that $|P|$ is as small as possible. From the minimality of P one can deduce that the circuit $C = P \cup \{e\}$ is isometric. Then the edge e' opposite to e in C must be in U_j. This contradiction proves (i).

Let A_j and B_j be the node sets of the two components of $(T, U - U_j)$. Using (5.2), one shows that for any path P in H,

$$|P| - |P \cap U_j| \ge \mathrm{d}^H(st) - \Delta(st), \tag{5.3}$$

where s, t are the ends of P, $\Delta(st) = 1$ if $|\{s,t\} \cap A_j| = 1$, and $\Delta(st) = 0$ otherwise. This easily implies (iii). Also (5.3) implies that the graph $H_j = H/U_j$, obtained from H by contracting the edges in U_j and then identifying parallel edges appeared, satisfies $\mathrm{d}^{H_j}(s't') = \mathrm{d}^H(st) - \Delta(st)$ for all $s, t \in T$, where s', t' are the images of s, t in H_j. Moreover, one shows that H_j is again a $K_{2,3}$-free frame (otherwise H violates (4.1)). This is used to prove (ii) as follows. Suppose that $\mathcal{A}$ contains pairwise crossing A_ℓ, A_q, A_r, and let $H' = (T', U')$ be $H/(U - U_\ell \cup U_q \cup U_r)$. Then H' is a $K_{2,3}$-free frame (by induction on p). The images U'_ℓ, U'_q, U'_r of U_ℓ, U_q, U_r are the orbits of H', and the corresponding subsets A'_ℓ, A'_q, A'_r of T'

are pairwise crossing. Furthermore, two dual paths with the edges belonging to different orbits share at most one 4-circuit (this easily follows from (5.2)). This implies that every dual path in H' has, at most, two 4-circuits, from which one can conclude that H' is C_6^+ (see Figure 6); a contradiction.

To see (iv), let $\mathcal{A}' \subset \bar{\mathcal{A}}$ consist of pairwise intersecting sets. We may assume that $\mathcal{A}' = \{A_1, \ldots, A_q\}$ and that no member of $\mathcal{A}'$ is a subset of another member. Then $A_i \cup A_j = T$ for any noncrossing $A_i, A_j \in \mathcal{A}'$. Let R_i denote the set of ends of edges of $\delta(A_i)$ that are contained in A_i. No edge in $\delta(A_1)$ belongs to another cut $\delta(A_i)$; therefore, if A_1 and $A_i (2 \le i \le q)$ are noncrossing, then $T - A_1 \subseteq A_i$ implies $R_1 \subseteq A_i$. Suppose that A_1 and some $A_i (2 \le i \le q)$ are crossing. Then there is a 4-circuit $C = v_0v_1v_2v_3v_0$ such that the edges v_0v_1 and v_2v_3 are in $\delta(A_1)$, while the edges v_1v_2 and v_3v_0 are in $\delta(A_i)$; let for definiteness $v_0 \in R_1 \cap R_i$. Since each $A_j \in \mathcal{A}'$ is not crossing at least one of A_1, A_i (as $\mathcal{A}$ is 3-cross-free), the above argument implies that $v_0 \in A_j$. Therefore $A_1 \cap \cdots \cap A_q$ is always nonempty. □

This theorem enables us to show that every $K_{2,3}$-free frame H is minimizable, as follows. By (i) and (iii) in Theorem 5.2, the above metric $m = \rho^{X_1} + \cdots + \rho^{X_p}$ is an extension of H, and by (ii) and the argument before the theorem, m is an optimal solution to (1.2). Note that the minimum cuts $\delta(X_1), \ldots, \delta(X_p)$ can be chosen so that in the family $\mathcal{X}$ consisting of the sets X_i and their complements $V - X_i$, any two members X and Y are intersecting only if $X \cap T$ and $Y \cap T$ are intersecting. Indeed, if this is not so for some X, Y, replace $\delta(X), \delta(Y)$ by $\delta(X - Y), \delta(Y - X)$; by standard submodularity arguments, $\delta(X - Y)$ is also a minimum cut separating $A = X \cap T$ and $T - A$, and similarly for $\delta(Y - X)$. A sequence of $O(p|V|^2)$ such ‘uncrossing operations’ updates $\mathcal{X}$ so as to satisfy the above property.

Now, to conclude that H is minimizable, it remains to show that m is a 0-extension. We prove the latter fact in a more general situation (for purposes of Section 6). Let us say that a graph $H = (T, U)$ is *strictly cut decomposable* if d^H is representable as the sum $\rho^{A_1} + \cdots + \rho^{A_p}$ of cut metrics on T, and the family $\{A_1, \ldots, A_p, T - A_1, \ldots, T - A_p\}$ satisfies the property in (iv) of Theorem 5.2. Thus, every $K_{2,3}$-free frame is strictly cut decomposable (in general, a strictly cut decomposable graph is not necessarily a frame). Assuming that H is strictly cut decomposable, that $\rho^{A_1} + \cdots + \rho^{A_p}$ is the corresponding representation of d^H, and that $m = \rho^{X_1} + \cdots + \rho^{X_p}$ is constructed as above, we assert that for each $x \in V - T$, there exists $s \in T$ with $m(xs) = 0$.

Indeed, let $\mathcal{X}'$ consist of all $X \in \mathcal{X}$ containing x. The members of χ' are pairwise intersecting, therefore, the members of $\mathcal{A}' = \{X \cap T : X \in \mathcal{X}'\}$ are so. Then the members of $\mathcal{A}'$ have a common element s. Obviously, none of the cuts $\delta(X_i)$ separates x and s, whence $m(xs) = 0$. Thus, m is a 0-extension.

One can see that m constructed this way is an optimal solution to (1.1) in the general case of strictly cut decomposable graphs. More precisely, observe that $c \cdot m = c\langle A_1 \rangle + \cdots + c\langle A_p \rangle$, and let m' be an arbitrary 0-extension of H to V. For $i = 1, \ldots, p$, let $X'_i = \{x \in V : m'(xs) = 0$ for some $s \in A_i\}$. Then $m' = p^{X'_1} + \cdots + \rho^{X'_p}$, whence $c \cdot m' = c(\delta(X'_1)) + \cdots + c(\delta(X'_p))$. Now $c(\delta(X'_i)) \ge c\langle A_i \rangle$ for $i = 1, \ldots, p$ implies $c \cdot m' \ge c \cdot m$, as required. Thus, (1.1) is reduced to $O(|T|)$ minimum cut computations, and we get the following result.

THEOREM 5.3. *For any fixed strictly cut decomposable $H = (T, U)$, problem (1.1) can be solved in strongly polynomial time.*

6. COMPLEXITY OF THE PROBLEM FOR NONMINIMIZABLE GRAPHS

Given a graph $H = (T, U)$, a set $V \supseteq T$, a function $c : E_V \to \mathbb{Z}_+$, terminals $s, t \in T$, and inner nodes $x, y \in V - T$, let $\tau(s, t, x, y)$ denote the minimum $c \cdot m$ among all 0-extensions m of H to V such that $m(sx) = m(ty) = 0$.

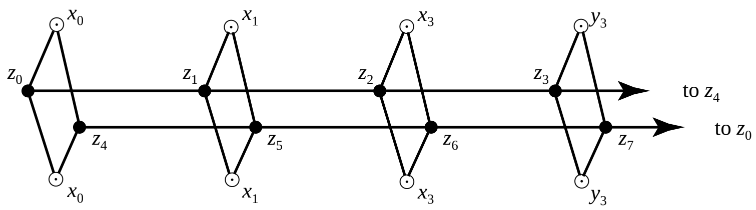

FIGURE 15.

Let $H = K_3$. The core of the proof in [4] that the 3-terminal cut problem is NP-hard is the construction of a 'gadget' (V, c) (or G_c) with specified terminals s, t and inner modes x, y which possesses the following properties:

(i) $\tau(s, t, x, y) = \tau(t, s, x, y) = \hat{\tau}$;
(ii) $\tau(s, s, x, y) = \tau(t, t, x, y) = \hat{\tau} + \Delta$ for some $\Delta > 0$;
(iii) $\tau(s', t', x, y) \geq \hat{\tau} + \Delta$ for all other pairs (s', t') in T, (6.1)

where $\hat{\tau}$ is the optimum value $\tau(H, c)$. Then the result is easily obtained by a polynomial transformation from the maximum cut problem (for details, see [4]).

We show that a gadget (V, c) satisfying (6.1) can also be constructed for any nonmodular or nonorientable graph H; this immediately implies Theorem 1.3 by use of a similar reduction.

Let us start with a nonorientable bipartite graph $H = (T, U)$. Choose an orientation-reversing dual cycle $D = (e_0, F_1, e_1, \ldots, F_k, e_k)$ in H, where $e_i = x_i y_i$, $F_i = x_{i-1}y_{i-1}y_i x_i x_{i-1}$, $x_k = y_0$ and $y_k = x_0$ (see the proof of Lemma 3.6). We denote y_i by x_{i+k} and take indices modulo $2k$. The desired gadget consists of the graph $G = (V, E)$ and weights $c(e)$, $e \in E$, where:

(i) $V = T \cup \{z_0, \ldots, z_{2k-1}\}$;
(ii) for $i = 0, \ldots, 2k-1$, z_i is connected by edges of weight $M \geq 2k + 2$ with both x_i and x_{i+k};
(iii) for $i = 0, \ldots, 2k-1$, z_i and z_{i+1} are connected by an edge of weight one.

The resulting graph G for $k = 4$ (when $|T| = 8$ and all $x_0, \ldots, x_{2k-1}$ are different) is drawn in Figure 15. We formally extend c by zero on $E_V - E$. Define $s = x_0$, $t = x_k$, $x = z_0$ and $y = z_k$. We assert that (6.1) is satisfied.

To see this, associate with each 0-extension m of H to V the mapping γ from $\{z_0, \ldots, z_{2k-1}\}$ to T by setting $\gamma(z_i) = x_j$ if $m(z_i x_j) = 0$; we say that z_i is *attached* to x_j and denote m by m^γ. Observe that if $\gamma(z_i) \in \{x_i, x_{i+k}\}$, then the contribution to the volume $c \cdot m^\gamma$ from the edges $e = z_i x_i$ and $e' = z_i x_{i+k}$ is equal to $c(e)m^\gamma(e) + c(e')m^\gamma(e') = M$ (since $m(x_i x_{i+k}) = 1$); otherwise the contribution is at least $2M$. Next, if z_i is attached to x_i (resp. x_{i+k}) and z_{i+1} to x_{i+1} (resp. x_{i+1+k}), then the edge $u = z_i z_{i+1}$ contributes $c(u)m^\gamma(u) = 1$, while if z_i is attached to x_i (resp. x_{i+k}) and z_{i+1} to x_{i+1+k} (resp. x_{i+1}), then the contribution is two (taking into account that $m(x_i y_{i+1}) = m(x_{i+1} y_i) = 2$ because H is bipartite). In view of $M > 2k$, these arguments show that $\hat{\tau} = 2kM + 2k$, and there are exactly two optimal 0-extensions, namely, m^{γ_1} and m^{γ_2}, where $\gamma_1(z_i) = \gamma_2(z_{i+k}) = x_i$ for all $i = 0, \ldots, 2k-1$. This implies (6.1) (i). Also the least-volume 0-extension $m = m^\gamma$ with $\gamma(z_0) = \gamma(z_k) = x_0$ establishes $m(z_j z_{j+1}) = 2$ for two instances of j, whence $\tau(s, s, x, y) = 2kM + 2k + 2$. Similarly, $\tau(t, t, x, y) = 2kM + 2k + 2$. This gives (6.1) (ii). Finally, unless both s', t' are in $\{x_0, x_k\}$, we have $\tau(s', t', x, y) \geq (2k-1)M + 2M \geq 2kM + 2k + 2$, yielding (iii).

This provides the NP-hardness for nonorientable bipartite graphs (note that by increasing the size of V by a factor of $O(k)$, one can get a gadget with all weights 0 and 1; so the problem is strongly NP-hard).

Now let $H=(T,U)$ be not modular (e.g. not bipartite). The construction of a gadget will rely on certain facts exhibited in (6.2)–(6.4) below. A node $v\in T$ that provides the equalities in (1.3) for a triple $\{s_0,s_1,s_2\}$ in T is called a *median* of this triple. Choose a median-less triple $\{s_0,s_1,s_2\}$ with $k=d(s_0s_1)+d(s_1s_2)+d(s_2s_0)$ minimum, where d stands for d^H. Obviously, $k\neq 0,1,2,4$. We say that a sequence $(v_0,v_1,\dots,v_r)$ of terminals (not necessarily a path in H) is shortest if $d(v_0v_1)+\dots+d(v_{r-1}v_r)=d(v_0v_r)$.

First, choose a shortest s_0–s_1 path $x_0x_1\dots x_p$, a shortest s_1–s_2 path $x_px_{p+1}\dots x_q$ and a shortest s_2–s_0 path $x_qx_{q+1}\dots x_k$ in H. Clearly each of $p,q-p,k-q$ is strictly less than $k/2$. Suppose that $q-p+1<k/2$. Then $\{x_0,x_{p-1},x_q\}$ has no median. Indeed, if it has a median u, then $u\neq x_0$ (since $d(x_qx_0)+d(x_0x_{p-1})=k-q+p-1>k/2\geq d(x_qx_{p-1})$). This implies $d(ux_p)+d(x_px_q)+d(x_qu)<k$, whence $\{u,x_p,x_q\}$ has a median v. The fact that each of (x_0,x_{p-1},x_p), (x_0,u,x_{p-1}) and (u,v,x_p) is shortest implies that (x_0,v,x_p) is shortest. Also (x_p,v,x_q) and (x_q,v,x_0) are shortest. Hence, v is a median for $\{s_0,s_1,s_2\}$; a contradiction.

Considering the circuit $x_0x_1\dots x_{k-1}x_0$ up to reversing and cyclically shifting and repeatedly applying the above argument, we conclude that any triple $\{x_0,x_{p'},x_{q'}\}$ with all p', $q'-p'$, $k-q'$ smaller than $k/2$ has no median. Thus, without loss of generality we may assume that

$$|d(s_is_j)-k/3|\leq 1 \qquad \text{for } 0\leq i<j\leq 2. \tag{6.2}$$

Second, we observe that

$$\text{for } i=0,1,2, \text{ if } z\in T \text{ belongs to a shortest } s_{i-1}\text{–}s_{i+1} \text{ path and satisfies } d(s_iz)\leq\min\{d(s_is_{i-1}),d(s_is_{i+1})\}, \text{ then } z \text{ is either } s_{i-1} \text{ or } s_{i+1}, \tag{6.3}$$

taking indices modulo 3. Indeed, suppose $z\neq s_{i-1},s_{i+1}$. Assuming $d(s_is_{i-1})\leq d(s_is_{i+1})$, take a median u for $\{s_i,s_{i-1},z\}$ (which exists because $d(s_is_{i-1})+d(s_{i-1}z)+d(zs_i)$ is, obviously, less than k). Then $u\neq s_{i-1}$, and the sequence (s_{i-1},u,s_{i+1}) is shortest. Therefore, the triple $\{u,s_i,s_{i+1}\}$ has a median v and, moreover, v is simultaneously a median for $\{s_0,s_1,s_2\}$; a contradiction.

Third, for $\{i,j,\ell\}=\{0,1,2\}$, if $d(s_is_j)<d(s_is_\ell)$, define s'_i to be a terminal adjacent to s_i and contained in a shortest s_i–s_ℓ path. Then $d(s'_is_\ell)=d(s_is_\ell)-1$; also $d(s'_is_j)=d(s_is_j)+1$, by (6.3) (with j in place of i). Moreover, arguing as in the proof of (6.2), observe that the triple $\{s'_i,s_j,s_\ell\}$ has no median. Now applying again (6.3) (with s'_i in place of s_i) and taking into account that $d(s_is_\ell)-d(s_is_j)=1$, we obtain the following:

$$d(s_is_{i-1})+d(s'_is_{i-1})=d(s_is_{i+1})+d(s'_is_{i+1})=:k_i, \text{ and if } z\in T \text{ belongs to a shortest } s_{i-1}-s_{i+1} \text{ path and is different from } s_{i-1} \text{ and } s_{i+1}, \text{ then } d(s_iz)+d(s'_iz)>k_i. \tag{6.4}$$

Now we are ready to construct the desired gadget (G,c); it is close to that in [4].

Let M be a rather large positive integer, and let $d_{i,j}$ stand for $d(s_is_j)$. The graph $G=(V,E)$ and weighting $c:E\to\mathbb{Z}_+$ are as follows:

(i) $V=T\cup X$, where X consists of six nodes x_0,x_1,x_2,y_0,y_1,y_2;

(ii) for $i=0,1,2$, there are four edges $s_{i-1}x_i,s_{i+1}x_i,s_{i-1}y_i,s_{i+1}y_i$ of weight M^2 and two edges s_ix_i,s_iy_i of weight M; in addition, if $d_{i,i-1}\neq d_{i,i+1}$, there are two edges s'_ix_i,s'_iy_i of weight M;

(iii) there are six edges of weight one forming the circuit $C=x_0x_1x_2y_0y_1y_2x_0$.

For all other pairs $uv\in E_V$, the weight $c(uv)$ is zero. As before, each 0-extension $m=m^\gamma$ is associated with a corresponding mapping $\gamma:X\to T$. Since M is large, the edges $s_{i-1}x_i$ and $s_{i+1}x_i$ of weight M^2 provide that if m^γ pretends to be optimal or nearly optimal (in the sense that $c\cdot m^\gamma-\hat{\tau}=O(1)$), then each x_i is attached to a terminal z contained in a shortest $s_{i-1}-s_{i+1}$ path, while (6.3), (6.4) and the edges s_ix_i and, possibly, s'_ix_i of weight M provide

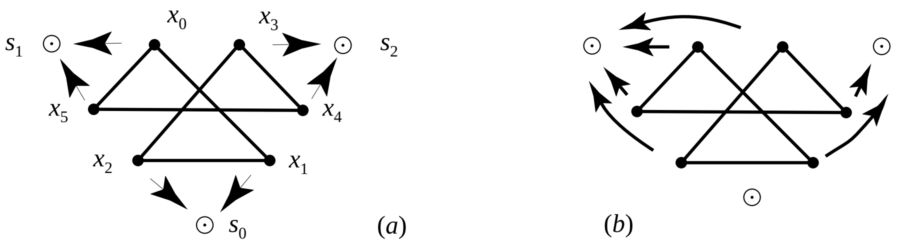

FIGURE 16.

that this z is either s_{i-1} or s_{i+1}. Similarly $\gamma(y_i) \in \{s_{i-1}, s_{i+1}\}$. Thus, it suffices to examine only those γ's for which $\gamma(x_i), \gamma(y_i) \in \{s_{i-1}, s_{i+1}\}$ for all i; such a mapping is called *feasible*. Note that for all feasible γ's the total contribution to the volume from the above big edges is the same. Let $\zeta(\gamma)$ denote the contribution from all edges of the circuit C.

It is convenient to denote y_i by x_{i+3}, define $s_{i+3} = s_i$, denote $e_i = x_{i-1}x_i$, and take indices modulo 6. In view of (6.2), one may assume that $d_{0,1} = d_{0,2}$. For the desired relations in (6.1) we take $s = s_1$, $t = s_2$, $x = x_0$, $y = x_3$. Let $m_1 = m^{\gamma_1}$, where $\gamma_1(x_i)$ is s_{i+1} for $i = 0, 2, 4$, and s_{i-1} for $i = 1, 3, 5$. This γ_1 attaches x to s and y to t, and

$$\zeta(\gamma_1) = \sum_{i=0}^{5} c(e_i)m_1(e_i) = d_{1,0} + 0 + d_{0,2} + 0 + d_{2,1} + 0 = k.$$

The γ_1 is illustrated in Figure 16(*a*). Similarly, $\zeta(\gamma_2) = k$ for the symmetric mapping γ_2, where $\gamma_2(x_i) = \gamma_1(x_{i+3})$.

One can see that for any feasible γ, there are at least three edges e_i with $m^{\gamma}(e_i)$ nonzero, and that m^{γ_1} and m^{γ_2} are the only 0-extensions with exactly three such edges. Therefore, for $\gamma \neq \gamma_1, \gamma_2$, we have $\zeta(\gamma) \geq 4b$, where $b = \min\{d_{0,1}, d_{1,2}, d_{2,0}\}$. This implies (6.1) whenever $4b > k$ (Figure 16(*b*)) illustrates the least-volume mapping attaching both x, y to s in case $d_{1,2} \leq d_{0,1} = d_{2,0}$). By (6.2), $k \in \{3b, 3b+1, 3b+2\}$. The only cases when $k \neq 0, 1, 2, 4$ and $4b \leq k$ are $k = 5$ and $k = 8$. For these cases we need another gadget. It is a '5-cycle' analog of the previous gadget, which we now construct.

For $i = 0, 1, 2$, take a shortest s_i–s_{i+1} path (letting $s_3 = s_0$), and let C be the concatenation of these three paths. Varying median-less triples $\{s_0, s_1, s_2\}$ in C and using (6.3), one can deduce that C is an isometric circuit. Let $t_0, \ldots, t_4$ be different nodes that follow in this order in C and satisfy $d_{i-1,i} + d_{i,i+1} \geq q$ for $i = 0, \ldots, 4$, where indices are taken modulo 5, $d_{j,j'}$ stands for $d(t_j t_{j'})$, $q = 2$ if $k = 5$, and $q = 3$ if $k = 8$ (when $k = 5$, C is just $t_0t_1t_2t_3t_4t_0$). Next, using (6.3) and (6.4) (with s_0, s_1, s_2 varied), one shows that for $i = 0, \ldots, 4$, there are nodes u_i, v_i in C (possibly $u_i = v_i$ in case $k = 8$) such that: (a) $d(u_i t_{i-1}) = d(v_i t_{i+1})$ and $d(v_i t_{i-1}) = d(u_i t_{i+1})$, and (b) no shortest t_{i-1}–t_{i+1} path has an intermediate node z with $d(u_i z) + d(v_i z) \leq d(u_i t_{i-1}) + d(v_i t_{i-1})$. Now form $G = (V, E)$ and c as follows:

(i) $V = T \cup X$, where X consists of ten nodes $x_0, \ldots, x_9$;
(ii) for $i = 0, \ldots, 9$, there are two edges $x_i t_{2i-1}$ and $x_i t_{2i+1}$ of weight M^2 and two edges $x_i u_{2i}$ and $x_i v_{2i}$ of weight M; letting $t_{j+k} = t_j$, $u_{j+k} = u_j$, $v_{j+k} = v_j$, and taking indices modulo 10;
(iii) for $i = 0, \ldots, 9$, there is an edge $e_i = x_{i-1}x_i$ of weight one.

The weights of edges in (ii) provide that a 0-extension m^{γ}, induced by a mapping $\gamma : X \to T$, is optimal or nearly optimal only if $\gamma(x_i) \in \{t_{2i-1}, t_{2i+1}\}$ for $i = 0, \ldots, 9$, in which case γ is called feasible. One may assume that $d_{0,1} = d_{4,0}$ and $d_{1,2} = d_{3,4}$. Set $s = t_1$, $t = t_4$, $x = x_0$

and $y = x_5$. Define $\gamma_1(x_i)$ to be t_{2i+1} for $i = 0, 2, 4, 6, 8$ and t_{2i-1} for $i = 1, 3, 5, 7, 9$, and define the symmetric mapping γ_2 by $\gamma_2(x_i) = \gamma_1(x_{i+5})$. Then $\gamma_1(x) = s$, $\gamma_1(y) = t$, and γ_1 maps the sequence $(x_0, \dots, x_9)$ to $(t_1, t_1, t_0, t_0, t_4, t_4, t_3, t_3, t_2, t_2)$, whence the contribution from the circuit $C' = x_0 \dots x_9 x_0$ amounts to k. Similarly, $\gamma_2(x) = t$, $\gamma_2(y) = s$, and the contribution from C' under γ_2 is k.

On the other hand, one can check that for any feasible γ, (a) each pair of consecutive edges of C' contributes at least one, and (b) unless γ is γ_1 or γ_2, there are at least two edges e_i of C' whose ends x_{i-1} and x_i are attached to terminals t_j and $t_{j'}$ with $|j - j'| = 2$. For such j, j', we have $d_{j,j'} \geq q$, and now (a) and (b) easily imply that for $\gamma \neq \gamma_1, \gamma_2$, the contribution from C' is at least $k + 1$. This provides (6.1) and completes the proof of Theorem 1.3. □

Note that there is a gap between the set of frames (for which the minimum 0-extension problem is polynomially solvable) and the set of orientable modular graphs (beyond which the problem becomes intractable). For example, the 1-skeleton Q_k of the k-dimensional cube is orientable and modular but not a frame for any $k \geq 3$, as it contains an isometric 6-circuit (the nodes of Q_k are represented as the binary k-vectors and the edges as the pairs of vectors that differ at exactly one component). Bandelt [2] proved that every modular but not hereditary modular graph includes Q_3 or Q_3^+ as an isometric subgraph, where Q_3^+ is obtained by adding to Q_3 one edge between nodes of distance three. Note that Q_3^+ is not orientable because it includes $K_{3,3}^-$. In contrast, each graph Q_k is strictly cut decomposable (see Section 5 for the definition), and therefore, problem (1.1) for Q_k is solvable in strongly polynomial time, by Theorem 5.3.

Yet, not every graph which is orientable and modular but not a frame is strictly cut decomposable. What is the complexity status of (1.1) for such graphs? In particular, let H be the union of a frame and a strictly cut decomposable graph whose intersection consists of a single node or a single edge. Is (1.1) for this H solvable in polynomial time?

Another open question: Can one give a ‘good characterization’ for the set of minimizable metrics μ?

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